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THERMAL BOUNDARY LAYER IN A GAS SUBJECT TO A TIME DEPENDENT PRESSURE

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ABSTRACT

An analysis of the thermal boundary layer in a gas subject to time dependent pressure variations has been made for the case in which the thermal conductivity is proportional to the temperature. Explicit expressions for the time dependent boundary layer profile, the displacement thickness and the heat transfer rate are presented.

Introduction

The heat loss from a gas undergoing time dependent pressure variations is required for the analysis of data from constant volume combustion bombs used to measure laminar flame speeds [1]. It is also of interest in connection with energy balance calculations for reciprocating compressors and internal combustion engines.

Boundary Layer Equations

To investigate this problem, we consider the idealized case of a perfect gas in contact with a plane wall of infinite thermal conductivity. We assume that radiation is negligible and that the pressure p throughout the thermal boundary layer is independent of the distance x from the wall and a function only of the time t. This requires that the transit time of a sound wave across the boundary layer be small compared to the characteristic time for changes in the pressure. Under these conditions the equations for conservation of mass and energy within the boundary layer are [2]

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \qquad (1)$$

and

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x}\right) = \frac{dp}{dt} + \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} \tag{2}$$

where ρ and T are the density and pressure of the gas, u is the x component of the gas velocity and c_p and k are the constant pressure specific heat and thermal conductivity of the gas.

Integrating equation (1) with respect to x at constant t, we obtain

$$(\partial\eta/\partial t) = (\rho/\rho_o)\, u = 0 \tag{3}$$

where ρ_o is the initial gas density and we have introduced the scaled space coordinate

$$\eta(x,t) = \int_o^x (\rho/\rho_o)\, dx' \tag{4}$$

Using equation (3) to eliminate u in equation (2) and transforming from x to η gives

$$c_p\rho_o \frac{\partial T}{\partial t} = \frac{\rho_o}{\rho} \frac{dp}{dt} + \frac{\partial}{\partial\eta} \frac{\rho k}{\rho_o} \frac{\partial T}{\partial\eta} \tag{5}$$

Using the equation of state

$$p = \rho RT \tag{6}$$

and the relation

$$dp/dt = (c_p p/RT_\infty)\, dT_\infty/dt \tag{7}$$

for isentropic compression of the gas external to the boundary layer, equation (5) may be further simplified to give

$$c_p\, \rho_o\, T_\infty \frac{\partial}{\partial t}\left(\frac{T}{T_\infty}\right) = \frac{\partial}{\partial\eta} \frac{\rho k}{\rho_o} \frac{\partial T}{\partial\eta} \tag{8}$$

Finally approximating the thermal conductivity by the expression

$$k(T) = K_o\, (T/T_o) = \rho_o c_p \alpha_o\, (T/T_o) \tag{9}$$

where α_o is the thermal diffusivity at the initial temperature and density and introducing the scaled time variable

$$s(t) = \alpha_o \int_o^t (p'/p_o)\, dt' \tag{10}$$

we obtain from equation (8) the simple equation

$$\frac{\partial}{\partial s}\left(\frac{T}{T_\infty}\right) = \frac{\partial^2}{\partial\eta^2}\left(\frac{T}{T_\infty}\right) \tag{11}$$

The solution of equation (11) which satisfies the initial condition

$$T\ (\eta,0) = T_\infty\ (0) \tag{12}$$

and boundary conditions

$$T\ (0,s) = T_o\ ,\ T\ (\infty,s) = T_\infty\ (s) \tag{13}$$

is

$$\begin{aligned} \frac{T}{T_\infty} &= 1 + \int_o^t \text{erfc}\left(\frac{\eta}{2\sqrt{s-s'}}\right) \frac{d}{dt'}\left(\frac{T_o}{T_\infty'}\right) dt' \\ &= 1 + \int_o^t \text{erfc}\left(\frac{\eta}{2}\left(\alpha_o \int_{t'}^t \left(\frac{p''}{p_o}\right) dt''\right)^{-1/2}\right) \frac{d}{dt'}\left(\frac{T_o}{T_\infty'}\right) dt' \end{aligned} \tag{14}$$

where

$$\text{erfc}\ (z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp\ (-y^2)\ dy \tag{15}$$

is the complimentary error function. The temperature ratio T_∞ / T_o may be obtained as a function of the pressure by integrating equation (7) and is given by the relation

$$T_\infty / T_o = (p / p_o)^{(1 - 1/\gamma)} \tag{16}$$

where $\gamma = c_p / c_v$ is the specific heat ratio.

Given a value for η and the pressure-time history p(t), equation (14) may be integrated either numerically or by approximate analytic technique to obtain $T(\eta,t)$. The value of the space coordinate x can then be determined from the equation

$$x\ (\eta,t) = \int_o^\eta (\rho_o / \rho')\ d\eta' = \int_o^\eta (T' / T_o)\ d\eta' \tag{17}$$

obtained by inversion of equation (4).

Displacement Thickness and Heat Transfer Rate

In most practical applications a complete solution for the time dependent temperature distribution in the thermal boundary layer is not needed and only the displacement thickness and wall heat transfer rate are required. These may easily be obtained using the results of the preceding analysis.

By definition the boundary layer displacement thickness is

$$\delta = \int_0^\infty \left(\frac{\rho}{\rho_\infty} - 1\right) dx = \frac{\rho_o}{\rho_\infty} \int_0^\infty \left(1 - \frac{T}{T_\infty}\right) d\eta \tag{18}$$

Substituting equation (14) into equation (18) and integrating over η we find

$$\delta = -2\left(\frac{\alpha_o}{\pi}\right)^{\frac{1}{2}} \frac{\rho_o}{\rho_\infty} \int_o^t \left(\int_{t'}^t \frac{p''}{p_o} dt''\right)^{\frac{1}{2}} \frac{d}{dt'}\left(\frac{T_o}{T'_\infty}\right) dt'$$

$$= 2\left(1 - \frac{1}{\gamma_u}\right)\left(\frac{\alpha_o p_o}{\pi p}\right)^{\frac{1}{2}} \int_{p_o}^{p} \left(\int_{t'}^t \frac{p''}{p} dt''\right)^{\frac{1}{2}} \left(\frac{p}{p'}\right)^{\left(2-\frac{1}{\gamma_u}\right)} \frac{dp'}{p} \tag{19a}$$

or integrating by parts

$$\delta = \left(\frac{\alpha_o}{\pi}\right)^{\frac{1}{2}}\left(\frac{p_o}{p}\right)^{\frac{1}{\gamma_u}} \int_o^t \left(\int_{t'}^t \frac{p''}{p_o} dt''\right)^{-\frac{1}{2}} \left(\frac{p'}{p_o} - \left(\frac{p'}{p_o}\right)^{\frac{1}{\gamma_u}}\right) dt' \tag{19b}$$

The corresponding heat loss from the gas per unit area of wall is given by

$$\dot{q} = k \left.\frac{\partial T}{\partial x}\right|_o = \frac{\rho k}{\rho_o} \left.\frac{dT}{d\eta}\right|_o \tag{20}$$

Integrating equation (8) and substituting into equation (20) then yields

$$\dot{q} = -\rho_o c_p T_\infty \frac{d}{dt} \int_o^\infty \frac{T}{T_\infty} d\eta \tag{21}$$

Using equations (18), (7) and (6) we also obtain

$$\dot{q} = c_p T_\infty \frac{d}{dt} (\rho_\infty \delta)$$

$$= \frac{d}{dt}(\rho_\infty c_p T_\infty \delta) - \delta \frac{dp}{dt}$$

$$= \frac{d}{dt}(\rho_\infty c_v T_\infty \delta) + p \frac{d\delta}{dt} \tag{22}$$

Integrating equation (22) yields

$$q = \rho_\infty c_v T_\infty \delta + \int_o^\delta p d\delta' \tag{23}$$

Equation (23) shows that the wall heat loss equals the sum of the thermal energy defect in the boundary layer plus the work done in compressing the boundary layer.

For the limiting case of a step increase in temperature at constant pressure such as that occurring when a thin flame impinges on a wall, equations (19) and 23 give the simple results

$$\delta = \frac{2}{\sqrt{\pi}}\left(\frac{T_\infty}{T_o} - 1\right)\sqrt{\alpha_o t} = \frac{2}{\sqrt{\pi}}\left(1 - \frac{T_o}{T_\infty}\right)\sqrt{\alpha_\infty t} \tag{24}$$

and

$$q = (\gamma/(\gamma-1))\ p\delta \tag{25}$$

For a rapidly rising pressure such as that occurring in constant volume combustion bombs, the principal contribution to the integral in equation (19) comes from the upper limit. In this case, the pressure may be approximated locally by an exponential function of the form

$$\frac{p'}{p_o} = 1 + \left(\frac{p}{p_o} - 1\right)\ \exp\left(-\frac{t-t'}{\tau}\right) \tag{26}$$

where

$$\tau = p/\ (dp/dt) \tag{27}$$

is a characteristic time for the pressure rise. Substitution of equation (27) into equation (20) then gives the displacement thickness

$$\delta = 2\left(\frac{\alpha_o\tau}{\pi}\right)^{1/2}\left(\frac{p_o}{p}\right)^{1/2}\left(\left(\frac{p}{p_o}\right)^{(1 - 1/\gamma)} - 1\right) K\left(\frac{p}{p_o}\right)\ , \tag{28}$$

where the function

$$K\ (z) = (1 - 1/\gamma)\int_1^z (1 - y^{-1})^{1/2} y^{-1/\gamma} dy \Big/ \left(x^{(1 - 1/\gamma)} - 1\right) \tag{29}$$

is essentially independent of γ and approaches 1 for $z \gg 1$ and $(2/3)\ \sqrt{z - 1}$ for $(z - 1) \ll 1$. The heat loss rate obtained using equations (22), (27) and (6) is

$$\dot{q} = \frac{p}{(\gamma-1)\tau}\left(\delta + \gamma p\,\frac{d\delta}{dp}\right) \tag{30}$$

Values of the functions K, $\delta/\sqrt{\alpha_o\ \tau}$ and $(p\ /\sqrt{\alpha_o\ \tau})d\delta/dp$ are given in Table 1 as a function of p/p_o for $\gamma = 1.4$. For other values of γ, one may coupute δ using the values of K and equation (28).

Table 1

Data for computing the boundary layer displacement thickness and wall heat transfer rates for $\gamma = c_p/c_v = 1.4$. For practical purposes K is independent of γ.

$\frac{p}{p_o}$	K	$\frac{\delta}{\sqrt{\alpha_o \tau}}$	$\frac{p}{\sqrt{\alpha_o \tau}} \frac{d\delta}{dp}$
1.1	.203	.006	.092
1.2	.278	.015	.119
1.5	.404	.046	.148
2.0	.510	.089	.152
3.0	.615	.148	.134
5.0	.705	.208	.100
10.0	.787	.261	.056
20.0	.842	.288	.022
50.0	.890	.292	- .008
100.0	.916	.282	- .021

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Nomenclature

Roman

c	specific heat
k	thermal conductivity
K	function defined by equation (29)
p	pressure
$\dot{q}$	heating rate per unit area
R	specific gas constant
s	scaled time defined by equation (10)
t	time
T	temperature
u	velocity normal to wall
x	distance from wall

Greek

α thermal diffusivity

γ specific heat ratio

δ boundary layer displacement thickness

η scaled distance defined by equation (4)

ρ density

τ characteristic time defined by equation (27)

Subscripts

o value at t = o

∞ value at $x = \infty$

v constant volume

p constant pressure

References

1. See e.g., M. Metghalchi and J.C. Keck, Combustion and Flame 38, 143 (1980), T.W. Ryan and S.S. Lestz, SAE paper 800103.

2. H. Schlichting "Boundary Layer Theory," p. 252, McGraw-Hill, New York, (1968).