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NATIONAL

Made in U. S. A.

THIS BOOK CONTAINS EYE-EASE PAPER

"Easy on the Eyes"

Nuclear Physics

H. Bethe Spring 1947

Lecture I 2/21/47

Our information about nuclei is derived from the following sources:

1. Statistics of existing nuclei
   a. Gamow Valley (N vs Z)
2. Ground state
   a. Energy - the mass of a nucleus is always less than the mass of its component protons & neutrons. The difference of these masses gives the binding energy of the nucleus
   b. Spin
   c. Magnetic Moment
   d. Electric quadrupole moment

3. Spectroscopy

a. Bound excited states of nuclei

The data available here are too few to be really useful.

4. Scattering

5. Disintegration (too complicated)

In atomic physics almost all important information was gained from the study of the simplest systems i.e. H-atom and Alkali atoms. In nuclear physics, we shall proceed similarly and study the simplest available systems.

The fundamental particles in the nucleus are the proton H and the neutron N. The systems N^2, H^2 and NH are available for study. NH, the dueteron, is the only one of

these systems which is a stable nucleus.

1. Statistics

For $A = 1$ to 242 there exists at least one (stable) nucleus for each A except $A = 5$

Possible nuclei with $A = 5$ are

$$He^5 \equiv He^4 + N$$
$$Li^5 \equiv He^4 + H$$

The fact that these nuclei do not exist is associated with the fact that He^4 is such a stable system

For $A = 8$ we have an anomalous situation. The most stable nucleus ought to be Be^8. However, Be^8 is not observed while Li^8, a beta active element, has been observed. The reaction postulated is:

$$Li^8 \xrightarrow{\beta} Be^8 \rightarrow 2He^4$$

This equation raises the question of stability.

2. Stability

a. Stability against disintegration into heavy particles. Half lives for heavy particle disintegrations are of the order of time it takes the particle to traverse the nucleus:

10^{-12} cm / 10^{9} cm/sec = 10^{-21} sec

Times of this order are obviously too short to be observed.

b. Stability against β disintegration. Nuclei do not like to emit β's. The half lives for β disintegration vary from 10^{-2} sec to 10^{16} sec. Hence a β active nucleus may be easily observed.

3 Collisions

Always result in heavy particles or gammas never β's. This is one argument against the existence of electrons in a nucleus.

Very short half lives are always observed for neutron emission or the emission of heavy charged particles from light nuclei. The half lives for the emission of charged particles from heavy nuclei may be quite long if the particles are below the potential barrier and have to "tunnel" thru.

Short $T_{1/2}$

$$Z^{A} \rightarrow Z^{A-1} + N$$

Will go if

$$M(Z^{A}) > M(Z^{A-1}) + M_N$$

$$Z^{A} \rightarrow (Z-1)^{A-1} + H$$
$$Z^{A} \rightarrow (Z-2)^{A-4} + He$$

These reactions go with a short $T_{1/2}$ if the difference in energy between the LHS and the RHS is greater that the height of the potential barrier, B. They go with long half lives if the energy difference is a great deal less than the barrier height.

References Gamow
Rasetti

2/24/47

I Stability

A reaction is energetically possible if:

$$M(Z^A) > M'(Z'^{A'}) + M''(Z-Z')^{A-A'}$$

If the reaction is energetically possible we can distinguish two types of instability:

a. Classical instability

$$M - (M' + M'') > B$$

The half lives associated with these processes are very short of the order of 10^{-21} sec. The chief example of this kind of instability occurs in fission:

$$U^{235} + N \rightarrow U^{236*} \rightarrow (2\,Pd)$$

The height of the potential barrier for this case is:

$$B = 46^2 e^2 / R_0$$

If we take $R \sim 2r_0 = 14 \times 10^{-13}$

$= 5 \cdot 2.8 \times 10^{-13} = 5\,a_{0N} = 5(e^2/mc^2)$

$$B = 46^2\, 5\left(\frac{e^2}{mc^2}\right) = 46^2\, \frac{mc^2}{5} = 46^2\, \frac{1}{5}\,\frac{1}{2}\, 10^6 \text{ ev}$$

$$B \sim 200 \text{ Mev}$$

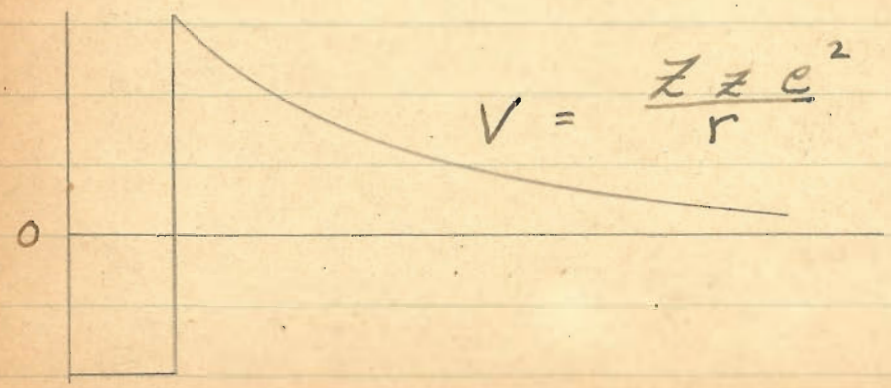
This is indeed approximately equal to the mass difference involved.

Ref Frisch & Peierls
Hahn & Strassman

b. Q.M. instability

Ref Gamow
London & Gurney

Suppose we have a potential of the type shown below (represents an nucleus)



Quantum mechanically it is possible for a particle (z) to get thru the barrier even though $E < V_{max} = B$ The probability of this is given by the expression

$$|\psi|^2 \sim e^{-\frac{2}{\hbar}\int \sqrt{2m(V-E)}\,dr}$$

The kinetic energy with which the particle is emitted, i.e. the K.E. at $r = \infty$, in the center of mass system is just E.

Obviously for any Zz there is a value of E which determines a practical limit of stability: The lower limit of observability for a nucleus is $\sim 1\ \mu s$. The upper limit of observability of radioactivity corresponds to a half life of 10^{14} yrs.

To compute the rate at which a nucleus decays, we must multiply $|\psi|^2$ by a quantity having the dimensions of a reciprocal time. There are arguments to show that this quantity should be of the order of 10^{21} sec^{-1}.

$$\text{decay rate} \sim 10^{21} e^{-\frac{2}{\hbar}\int \sqrt{2m(V-E)}\,dr}$$

The decay rate is an extremely sensitive function of E. To illustrate this we take the case of:

ThC'	10^{-7} sec	9 Mev α's
Th	10^{10} yrs	4 Mev α's

Here the height of the potential barrier is ~ 30 Mev

There is no nucleus energetically unstable against proton emission. Energetically the most favorable particles for emission are:

α, C^{12}, O^{16}. Due to the dependance of the potential barrier height of Zz, the α particle is emitted with a much greater probability than the others. Fission is a very unlikely process, but even so the half life for fission of U^{236} varies from $\sim 10^{14}$ yrs in the ground state to 10^{-18} sec in an excited state of 6 Mev.

II Packing Fraction

$$f = 10^4 \frac{M-A}{A}$$

$$H = 1.00812$$
$$n = 1.00893$$
$$\alpha = 4.00390$$

The binding energy of a nucleus is given by:

$$BE = -M(Z^A) + ZH + (A-Z)n$$

For nuclei in the neighborhood of the minimum

$$BE = -(-8)A + 81Z + 89(A-Z)$$
$$= 93 \times 10^{-4} A$$
$$= 8.7A \text{ Mev}$$

From this we see that the binding of a nucleus is proportional to A. In the atomic case the binding energy is proportional to Z^2. It is thought that in the nuclear case the forces <u>saturate</u>. Each particle can interact strongly with only 3 other particles (α part.)

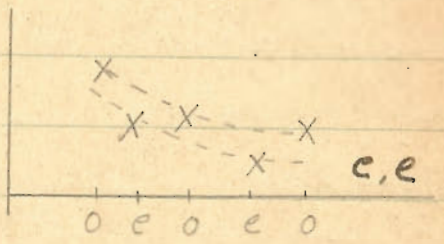
The Even-Odd Fluctuations

for particles in a nucleus are given in the table below:

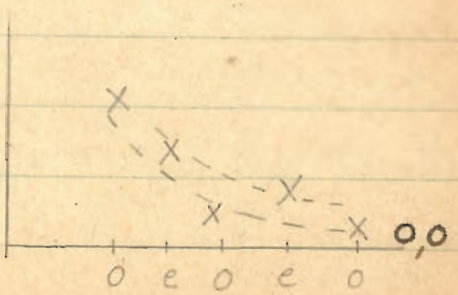
Z \ A	Even	Odd
Even	preferred	middle
Odd	abhored	middle

This behavior is reflected in the tightness with which the nuclei are bound

	A	
^{12}Mg	23	23.000
	24	23.992 5
	25	24.993 8
	26	25.989 8
	27	26.992 8



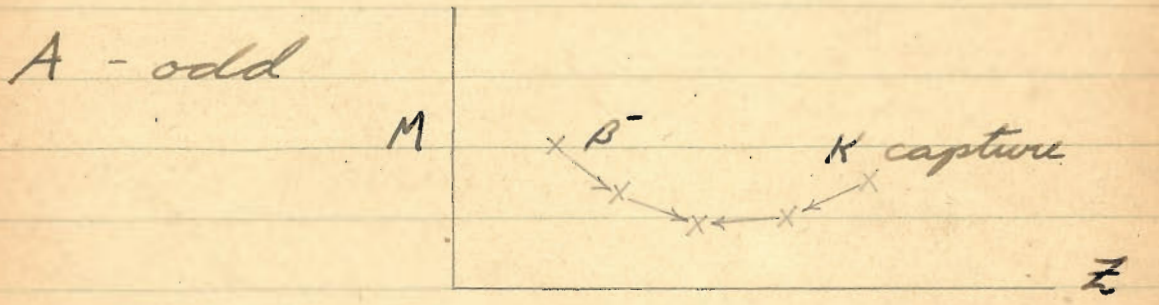
^{13}Al	25	24.9981
	26	25.9929
	27	26.8899
	28	27.9903
	29	28.9893



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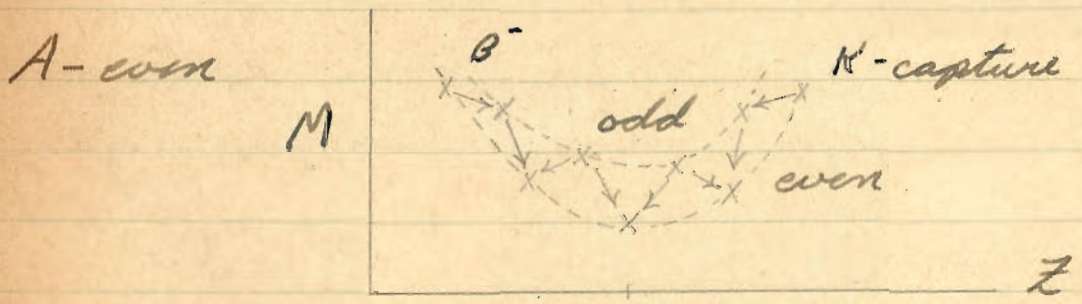
I Stability of nuclei

Consider the plots of mass versus charge below:



For odd A, we have only even-odd (odd-even) neutron proton configurations. The even-odd odd-even cases are about equally likely. The mass vs Z curve looks as pictured above and it will be seen that in general one would expect to find only one stable nucleus for each odd A. This is the case except for: $Rb^{87} \rightarrow Sr^{87} + \beta$; In^{113}, Cd^{113}; and Os^{187}, Re^{187}. It

is assumed that even these nuclii are not all stable but undergo either β-disintegration or K-capture with extremely long half lives.



For A even the mass vs Z curves are shown above. It is obvious that in this case we the possibility of several stable nuclii of even Z. All nuclii of odd Z should be unstable. This is reflected in the fact that nuclii of even Z have many isotopes and are more abundant since there are many ways of obtaining a

given element. Exception to this rule are H^2, Li^6, B^{10}, N^{14} $K^{40} \rightarrow Ca^{40} + \beta$.

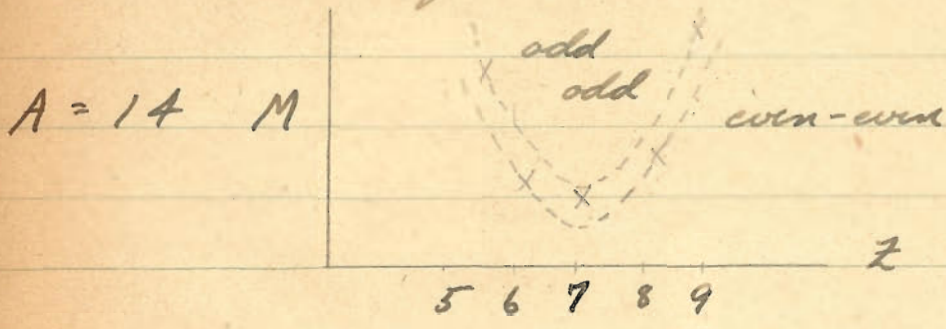
There is one stable isotope of each element below radioactive elements for all Z except 43 and 61. The neighboring elements occupy lower energy states.

Ratio of Neutrons to Protons

$$\frac{A-Z}{Z}$$

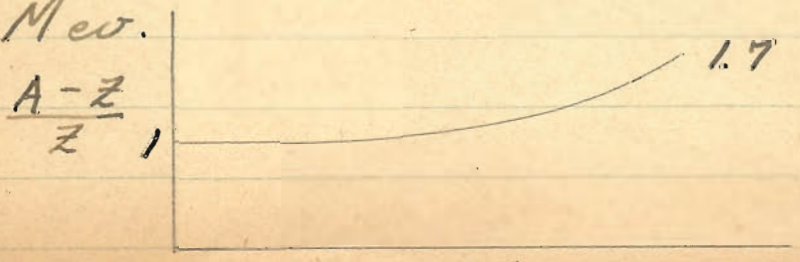
For light elements $A - Z \approx Z$. From this fact one can conclude a symmetry in the nuclear forces $n-n \sim p-p$. There is a preference for $n-p$ pairs. Effectively the $n-p$ force is greater than the $n-n$ or $p-p$ force. The preference for equal numbers of protons and neutrons in the nucleus explains the.

exception H^2; Li^6 B^{10} N^{14} to the even-odd rule. This is illustrated in the plot below



Note that due to the sharp minimum the most stable nucleus is $Z = 7$.

For the heavy elements the ratio of n to p increases due to the electric static repulsion between protons. The energy associated with the coulomb repulsion of two protons is of the order of $e^2/c^2/mc^2 \sim mc^2$ = .5 Mev.



Semi Empirical treatment of packing fraction curve.

The total binding energy of a nucleus is proportional to the volume of the nucleus minus a term proportional to the surface area of the nucleus to correct for the fact that nuclii at the surface do not interact as strongly with all other nuclii minus a term to correct for the coulomb repulsion between protons:

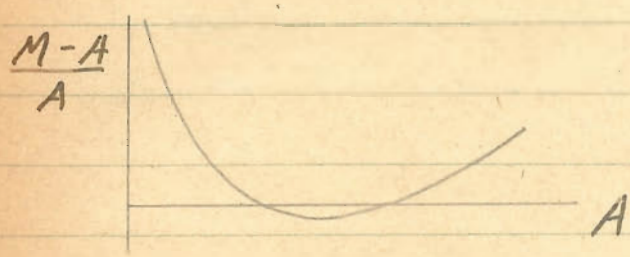
$$B.E. = -Vol \cdot B + Surf \cdot C + \frac{Z^2}{R} \cdot D$$

$$= -A \cdot B + A^{\frac{2}{3}} \cdot C + A^{\frac{5}{3}} \cdot D$$

$$\frac{B.E.}{A} = -B + CA^{-\frac{1}{3}} + DA^{2/3}$$

This equation gives a curve having

the form of the packing fraction curve.



In addition to the terms considered above, two more terms may be added, one to account for the preference a nucleus has for equal numbers of proton and neutrons and one to account for the even-odd preferences.

$$F\frac{(A-2Z)^2}{A} + G\begin{pmatrix}2\\1\\0\end{pmatrix}\begin{matrix}\text{odd-odd}\\\text{even-odd}\\\text{even-even}\end{matrix}$$

The value of the constants in the above formula are determined by experiment.

Using the above formula one can find the most stable Z for a given A.

Atomic Masses

For convenience atomic rather than nuclear masses are used in energy calculations

β^- $M_a(Z^A) > M_a(Z+1)^A$

K cap. $M_a(Z^A) > M_a(Z-1)^A$

β^+ $M_a(Z^A) > M_a(Z-1)^A + 2M_e$

Masses are determined from:

a. Mass Spectrograph data

b. Disintegration data

c. β-disintegration data

Mass for light nuclei are pretty good (5 decimal places)

Better mass determinations are needed for heavy nuclei

Ref Bethe R.M.P. 8 · 82 36

9 69 37

9 245 37

3/3/47

Determination of Size of the Nucleus.

1. The size of radio active nuclii may be determined from the formula for the penetration of the potential barrier.

$$\frac{1}{T} = \frac{1}{T_0} e^{-\frac{2}{\hbar}\int_R^{r_0}\sqrt{2m(V-E)}\,dr}$$

where $V = \frac{Z z e^2}{r}$, R is the nuclear radius and r_0 is defined by $V(r_0) = E$. T_0 is taken as 10^{-21} sec. If the half life T is measured the nuclear radius may be determined. This determination is insensitive to T. Values of the nuclear radius for the radioactive series obtained by this method are: $R \sim 9 \times 10^{-13}$ cm. A slight decrease in R with A is noted over the range of active nuclei

$A = 238 \rightarrow 210$. This is in accord with theory. The method is due to Gamow (Rasetti).

In part B of Bethe's nuclear physics article in RMP, there is a derivation of the nuclear radius based on the spacing of the energy levels in the atom. This derivation gives radii of the order of 10^{-7} cm. It is incorrect because the change in the level spacing with increasing energy was neglected.

2. The size of nuclei in the middle of the periodic table may be deduced from the probability of reactions induced by charged particles bombarding the nucleus. At high Z, the energy necessary to penetrate the potential barrier is excessive.

At low Z, the reactions can not be interpreted.

The Rochester group did a lot of work on the p-n reaction. This reaction is easily detected since it leads to a radioactive isobar of the original nucleus. The radii of nuclei of Z from 20 → 40 are:
$R \sim 5.6 \times 10^{-13}$ cm.

Segre used this method on heavy nuclii (notably Bi) The reaction studied was α-n.

3. The size of light nuclei may be determined from the energy difference between mirror nuclei. The mirror image of a nucleus having Z protons and Z+1 neutrons is one having Z+1 protons and Z neutrons. If the n-n force is assumed

equal to the p-p force, the nuclear binding energy of two such nuclei should be the same except for the coulomb repulsion between the protons. If on proton is added to a nucleus of Z proton the difference in the binding energy will be:

$$Z e^2 / r_{av}$$

The nuclear forces tend to keep the nuclear density constant. We assume a uniform distribution of the protons in the nucleus. Under these conditions the average distance between the added proton and all the other protons, r_{av}, equals $\frac{5}{6} R$. Where R is the nuclear radius. Hence:

Diff in B.E. = Diff in mass

$$= \frac{6}{5} \frac{Z e^2}{R} + (H - n) : -.75 \text{ Mev}$$

Possible mirror nuclei are:

$$^{2}He^{3} \xleftarrow{\beta^-} {}^{1}H^{3}$$
$$^{4}Be^{7} \rightarrow {}^{3}Li^{7}$$
$$^{5}B^{9} \rightarrow {}^{4}Be^{9}$$
$$^{6}C^{11} \rightarrow {}^{5}B^{11}$$
$$^{7}N^{13} \rightarrow {}^{6}C^{13}$$
$$\ldots \rightarrow \ldots$$
$$^{16}S^{31} \rightarrow {}^{15}P^{31}$$

Let us consider as an example the reaction

$$^{7}N^{13} \xrightarrow{\beta^+} {}^{6}C^{13} + 2m_e + 1.20 \text{ Mev}$$

$$^{7}N^{13} - {}^{6}C^{13} = \frac{6}{5}\frac{Ze^2}{R} - .75 \text{ Mev} = 2.22 \text{ Mev}$$

$$\frac{6}{5}\frac{Ze^2}{R} = 5.9 \text{ electron masses}$$

$$R = \frac{6 \cdot 6}{5 \cdot 5.9}\frac{e^2}{mc^2} = 1.22\frac{e^2}{mc^2}$$

$$R \sim 3.4 \times 10^{-13} \text{ cm}$$

From a study of mirror nuclei we can deduce $R \sim Z^{\frac{1}{3}} \sim A^{\frac{1}{3}}$ and hence $A \sim$ volume.

Let us check a heavy nucleus against a light one:

$$\frac{A'}{A} = \frac{220}{13} = 17$$

$$\left(\frac{R'}{R}\right)^3 = \left(\frac{9}{3.4}\right)^3 = 19 .$$

The reaction $He \leftarrow H$ goes in the opposite direction, $Be \rightarrow Li$ goes by K-capture. $^5B^9$ is not stable but its energy may be deduced from the threshold energy of the reaction $Be(p\,n)$. This method is often used to establish the mass differences between nuclei.

4. The size of any nucleus may be determined from the fast neutron cross-section. The wave length of the neutron should be small compared to

the nuclear radius. 20 Mev neutrons have a λ of $\sim 10^{-13}$ cm. The cross section of the target nucleus will be

$$\sigma \sim \pi (R+a)^2$$

where a is a constant associated with the range of the neutron forces.

ref. { Amaldi
Graham & Seaborg

3/7/47

I Nuclear Size

The size of a nucleus is given quite accurately by the relation

$$R = 1.5 \times 10^{-13} A^{\frac{1}{3}} \text{ cm.}$$

Shaw has measured the total x-section of nuclei for 25 Mev neutrons. Since the total x-section

includes the shadow effect it is twice the geometrical x-section.
Ref PR 1945

The nuclear case differs from the atomic case in that all measurements of nuclear radii agree quite well, whereas in the atomic case different methods of measurement give different radii. The reason for this is obviously that the atomic radius is not well defined.

II Spin, Statistics & Magnetic Moment.

a. Spin

Nuclei of even A have integral spin.
Nuclei of odd A have half integral spin.
The proton and the electron have spin = 1/2
If we think of the spin in a given direction has being directed positively or negatively,

then the spins add algebraically.

Suppose a nucleus were compounded of A protons and A - Z electrons. The total number of particles would be 2A - Z. We see that if Z even we should have integral spin and if Z odd half integral spin - independent of A. This is contrary to fact and is another reason for believing there are no electrons in the nucleus.

All nuclii of even A and even Z have spin zero. Pairs of protons and neutrons sum to give zero spin. This phenomena is undoubtly associated with the preference the nuclii have for even - even configurations.

Spins range from 0 to $^{9}/_{2}$. Some interesting cases follow:

$^{9}/_{2}$: Kr^{83}, In^{113} In^{115} Bi^{209}

4: $K^{40} \rightarrow Ca^{40} + \beta$ Long life due to large spin change.

1: Odd Z Even A, H^2, Li^6, B^{10}, N^{14}, Be^{10} may have a large I.

As in the atomic case, the orbital motion contributes to the total spin. We shall use I to denote the total angular momentum of the nucleus in units of $\hbar$ (spin)

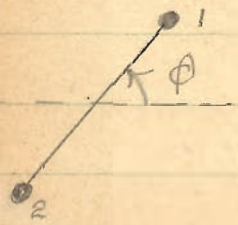
Spin may be determined from Hyperfine spectra, Band spectra intensities, magnetic resonance methods. Bethe R.M.P. part A

b. Statistics

Consider a diatomic molecule, say N_2. A particle obeys Fermi statistics if the interchange of two such particles in the diatomic molecule causes the wave function to change sign. A particle obeys Bose

statistics if the interchange leaves the wave function unchanged. Nuclei of odd A, electrons and protons obey Fermi statistics. It follows that if we interchange all the particles of an N_2 molecule (an even number of interchanges) this must leave the wave function unchanged and hence the neutron must also obey Fermi statistics. Nuclei of even A obey Bose statistics as implied above.

The type of statistics a nucleus obeys may be determined by a study of the rotational band spectra.



The rotational wave function is: $e^{im\phi}$. Interchanging particles 1 & 2 increases ϕ by π. The new wave

function becomes $(-1)^m e^{im\phi}$.
Hence, if the nucleus obeys Fermi statistics only odd rotational states will be present. If it obeys Bose statistics only even rotational states will be present.

It follows from all this that particles of half integral spin obey Fermi statistics and those of integral spin Bose statistics.

c. Magnetic Moment.

The magnetic moment is an axial vector parallel to the spin vector. The magnetic moment may be measured by magnetic resonance (Rabi). This method gives the gyro-magnetic ratio $\mu / I = g$. Hyperfine structure gives the product of the electronic wave function and

the magnetic moment. Purcell PR 69 37 and Bloch PR 69 127 have developed improved methods of measuring spin.

Nuclear m.m.'s are measured in units of $he/2Mc$. Some magnetic moments of interest are:

N	$-1.9103 \pm .001$	n.m.
	old $(-1.935 \pm .02)$	
H^1	$2.7896 \pm .0008$	
used as a calibration		
H^2	$.8565 \pm .0003$	
H^3	$1.067 \cdot H^1$	Theory 2.68 n.m.
He^3		Theory -1.48
He^4	0.000	
Li^6	.8213	$\sim He + H^2$
Li^7	3.2532	
In^{113}	5.48	
In^{115}	5.48	
Au	.23	

No nuclei have electric dipole moments. Quadrapole moments exist only if the spin ≥ 1
H^2, Li^6, Li^7 etc.

2/10/47

I Elementary Nuclei

A. Proton

Charge $+e$

Mass 1.008123 (938.1) Mev

Spin $1/2$

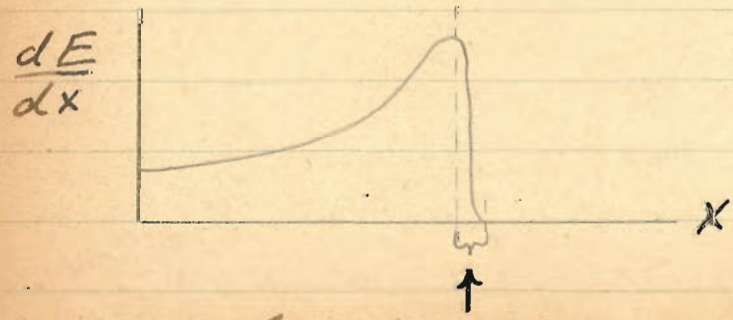
Mag. M. 2.7896 $\frac{e\hbar}{2Mc}$

Ref PR 56, 728 ; PR 69, 127
PR 69, 37

Ionizes

Bragg curve:

Ref: Bethe RMP C
Smith PR 71 32



Fraction of percent of range

The rate of energy loss for a proton is given by:

$$\frac{dE}{dx} \sim - \frac{NZe^2}{V^2} \ln(\ )$$

where N = no. atoms / cc
Z = charge of atoms
V = velocity of proton

Table of ranges and energy loss rates for protons

E-Mev	Range-cm	$\frac{dE}{dx}$ Kev/cm air	$\frac{dE}{dx}$ (electron
1	2.3	300	2.5
5	34	90	
10	115	51	
15	230	36	
50		13.5	
100	7100	8	2.2

B. Neutron

Charge 0

Mass 1.00893

The mass of the neutron has been

determined from the photo-disintegration of the deuteron.

$$H^2 + \delta \rightarrow p + n + T$$

The KE, T, is assumed equally divided between proton and neutron. The energy of the proton can be determined from its range. If the δ energy, (ThC" δ = 2.623 Mev) and the masses of the proton and deuteron are known, the mass of the neutron can be determined from the relation:

$$B.E. = H^2 - (p+n) = \delta - 2T_p$$

$T_p \approx 217 \pm 6$ kev. The B.E. of the deuteron turns out to be:

$$B.E = 2.189 \pm .035 \text{ Mev}$$

Ref: Stetter ZfP 110, 214
Bethe PR 53, 314

The neutron mass has been verified by p-n reactions. Consider

$$B^{11} + p + T \rightarrow n + C^{11} \rightarrow$$
$$n + B^{11} + e^{+} + .95\ Mev + m_0 c^2$$
$$m_0 = 0$$

The energy balance is:

$$p + T = n + 2m_e c^2 + .95\ Mev$$

Spin. The spin of a neutron is probably 1/2 (possibly 3/2)
Mag. M. $-1.9103 \pm .0012$ The minus sign means that the direction of the m. m. is opposite to that generated by a spinning positive charge.

Ref: Arnold + Roberts PR 70, 766

No ionization to speak of. 1 i p in 20-40 cm due to interaction of

magnetic moment for "fast Mev" neutrons. Secondary ionization by recoil particles. Range in a solid. 2-10 cm.

C. Deuteron

Charge	$+e$
Mass	2.014108
Spin	1
Mag. M.	$.8565 \pm .0004$

Knowing the m.m. of the p, n, & d, we can arrive at a probable value for the spin of the neutron. Bethe has tried the following possibilities:

Ground State of deuteron Calc.

S_N	m.m	S	L	μ
$3/2$	+	1	1	+ 2
$1/2$	+	0	1	.5
$-1/2$	-	0	1	.5
$-1/2$	-	1	0	.88

The last value of u tabulated agrees quite well with the measured value of u for the deuteron. The agreement is still better if we note that the ground state contains some "$L=2, S=1$" state.

3/17/47

I Quadrapole Moment of Deuteron

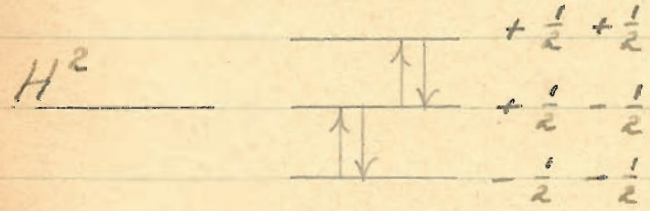
Since the deuteron has a quadrapole moment, we conclude that the ground state of the deuteron is not spherically symmetric. The quadrapole moment is given by:

$$\left[\frac{3}{2}Z^2 - \frac{1}{2}r^2\right]_{av} = Q$$

where the Z axis is taken parallel to the spin of the deuteron. r is distance from neutron to proton.

The quadrapole moment of the deuteron may be determined from the R.F. absorption spectrum of a D_2 molecule in a magnetic field.

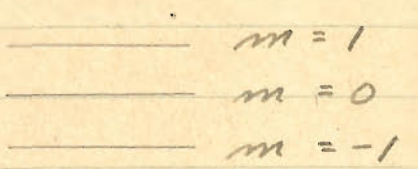
Consider the case of an H^2 molecule in a magnetic field. The energy levels will be split. If we apply an R.F. field having the Larmor frequency we can induce transitions as indicated below:



$$\nu_0 = \frac{eH}{2Mc}g$$

In the case of the deuteron molecule the interaction of the quadrapole moment with the field of the rest of the molecule complicates the energy situation.

We consider for simplicity only one of the deutrons of the D^2 molecule The deutron is a magnet of spin 1. In a mag. field the levels will be split as follows:



We have still to consider the effect of the total rotational quantum number, j.

In the case $j=0$ the axis of the molecule is oriented at random with respect to the magnetic field and there is no net effect on the energy levels.

In the case $j=1$ there will be a net magnetic moment due to rotation and this will result in further splitting of

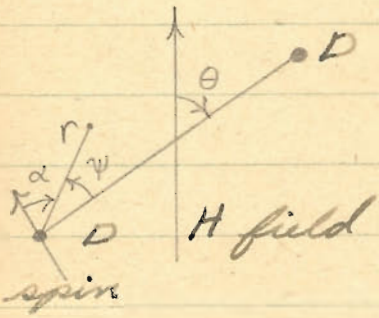
the energy levels

$m_j = 1$ $\quad \sin\theta\, e^{i\phi}$

m ——— $m_j = 0$ $\quad \cos\theta$

$m_j = -1$ $\quad \sin\theta\, e^{-i\phi}$

The angles are as in the following diagram



The energy levels associated with the rotational wave function K are:

$$E = Q \int \frac{P_2(\cos\alpha)}{r^3} P(r;\alpha)\, d\tau$$

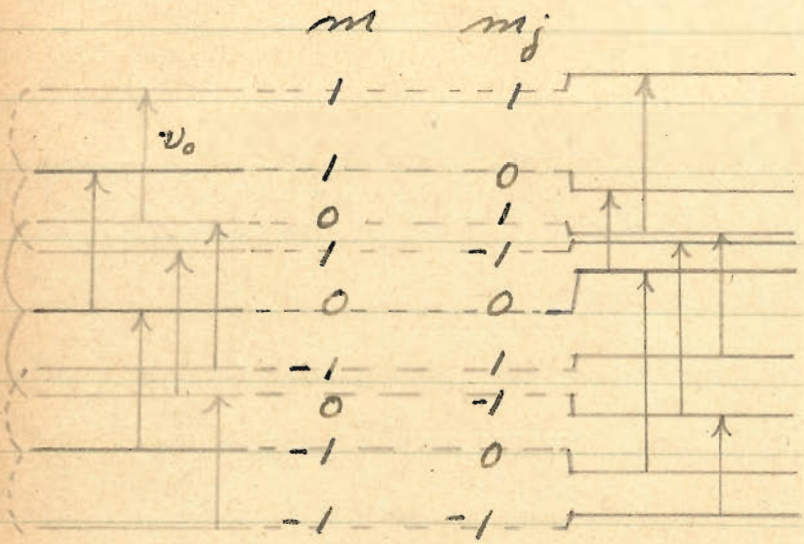
If we use the molecular axis as a reference

$$E = Q\, q\, \frac{3m^2 - S(S+1)}{(2S-1)(2S+3)} \frac{3m_j^2 - j(j+1)}{(2j-1)(2j+3)}$$

where $q = \left\langle \frac{P_2(\cos\psi)}{r^3} \right\rangle_{av} = \int \frac{P_2(\psi)}{r^3} P(r,\psi)\, d\tau$

4th

What happens on application of a magnetic is shown below:



Transitions induced by application of RF field.

On application of the quadrapole effect, the energy levels will be shifted according to the formula for E above:

$S = 1, \; j = 1$

Shift: δ for $|m| = |m_j| = 1$

-2δ for $|m| = 1 \;\; |m_j| = 0$

4δ for $m = m_j = 0$

where $\delta = Q_q$ In the case of a quadrapole moment instead

of one transition at ν_0 we have four transitions

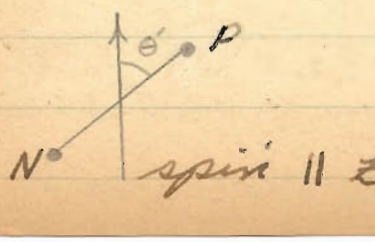
$$\nu_0 \pm 3\delta \quad \nu_0 \pm 6\delta$$

A knowledge of δ gives Qq. To compute Q we must know q. This requires a knowledge of the electronic wave function at the nucleus (because of the $\frac{1}{r^3}$ factor). The contribution of the other nucleus is easy to compute.

The value of Q is 2.73×10^{-27} cm^2 $\pm$ 2%. The size of the deuteron is $\bar{r}^2 \sim 10 \times 10^{-26}$ cm^2. Hence the asymmtry of the deuteron is about 3% of $\bar{r}^2$.

The wave function for the deuteron is:

$$\chi = u(r) + v(r) P_2(\cos\theta')$$



We can view this as a combination of S and D states. The prob of the S state is $\int u^2 d\tau$ The prob of the D state is $\int v^2 P_2^2 d\tau$. A calculation of this latter integral gives about .04.

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I We saw last time that the wave function for the deuteron in the ground state could be represented as the sum of the S and D states. The fraction of D state being about 4% from theory.

$$\psi = u(r) + v(r) P_2(\cos\theta)$$

We can determine the fraction of D state experimentally from the magnetic moments of the neutron, proton and deuteron.

$\mu_N = -1.9103$

$\mu_p = 2.7896$

If the ground state of the deuteron were spherically symmetric the resultant magnetic moment would be $\mu_{H^2} = .8793$

Experimentally $\mu_{H^2} = .8565$

$\mu_{H^2 S} - \mu_{H^2 X} = .0228$

If p is the probability of the D state, $1-p$ is the probability of the S state. The magnetic moment becomes

$$\mu_{H^2} = p\mu_D + (1-p)\mu_S$$

Due to the nature of the operators involved there is no interference term.

For the D state the spins are

$$\begin{cases} S = S_N + S_p = 1 \\ L = 2 \\ I = 1 \end{cases}$$

The resultant magnetic moment will be determined as follows:

$$(g_L \underline{L} + g_S \underline{S}) \cdot \underline{I} / I^2 = g_r$$

$$2\underline{L}\cdot\underline{I} = L^2 + I^2 - \underbrace{(\underline{L} - \underline{I})^2}_{\underline{S}^2}$$

The eigenvalues of the above operators

$$L \cdot I = \tfrac{1}{2}\{L(L+1) + I(I+1) - S(S+1)\}$$

$$g_r = \tfrac{1}{2}(g_L + g_S) + \tfrac{1}{2}(g_L - g_S)\,\frac{L(L+1) - S(S+1)}{I(I+1)}$$

This is the Landé formula. In our case $S = 1$, $L = 2$, $I = 1$ (D state)

$$g_r = \tfrac{3}{2} g_L - \tfrac{1}{2} g_S$$

Classically μ_L for a single particle is $-2\frac{e}{Mc}$, hence $g_L = 1$. In the deuteron, the angular momentum is shared equally by proton and neutron, hence $g_L = 1/2$

$$g_r = .750 - \tfrac{1}{2} g_S$$

$$\mu_{H^2} = .75\,p + (1 - \tfrac{3}{2}p)\,8793$$

$$\mu_{H^2} = 8793 - \underbrace{\tfrac{3}{2} p (8793 - \tfrac{1}{2})}_{.0228}$$

$$\tfrac{3}{2} p = .0228 / 1.379 = .06$$

$$p = 4\%$$

This is in excellent agreement with the results of Rarita and Schwinger 3.9%. The agreement is better than the theory. One would expect a difference of a few .1% due to the Binding of the deuteron.

II Two Body Problem. The only problem which can be solved exactly is the two body problem. We shall study the following cases:

1. Central Forces. These are the only forces for which there is good experimental evidence
2. Tensor Forces
3. Saturation of Forces

Binding energy considerations

# part.		BE	# pairs	
2	D	2.18 Mev		1
3	H^3	8	$3 \cdot 2/2$	3
4	He^4	27	$4 \cdot 3/2$	6

The kinetic energy is ~ to the number of particles. The potential energy is proportional to the number of pairs of particles:

$$E_D = 2T + V$$
$$E_T = 3T + 3V$$
$$E_\alpha = 4T + 6V$$

If we assume E_D small and negative, we find by comparing this qualitative table with the table of BE's that the potential well must be very deep and narrow. If the well were not narrow then the energy levels would lie deep in the well and $|V|$ would not be large compared with $|E|$. Keeping BE of H^2 constant adjust the dimensions of the well to give

the BE of H^3.

Ref PR '33 or '34

Thomas PR 36

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I Nuclear Potential

Assume $V = V(r)$, L = orbital angular momentum. The lowest energy level has a wave function with no nodes.

$\therefore \quad L = 0$

$\psi = u(r)/r$

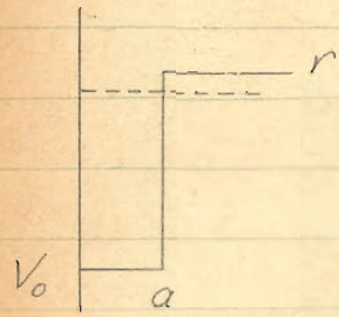
$u(r)$ has no zeros except at the origin.

$$\frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}(E - V(r))u = 0$$

$2\mu \approx$ mass of proton or neutron in case of deuteron.

Assume $|V(r)|_{av} >> B.E = \epsilon = -E$

A Square Well.



$$V(r) = -V_0 \quad r < a$$
$$V(r) = 0 \quad r > a$$
$$E = -\epsilon$$

$$r < a \quad \frac{d^2u}{dr^2} + \frac{M}{\hbar^2}(V_0 - \epsilon)u = 0$$

$$r > a \quad \frac{d^2u}{dr^2} - \frac{M\epsilon}{\hbar^2}u = 0$$

$$r < a \quad u = A \begin{matrix} \sin \\ \cos \end{matrix} Kr \qquad : K = \frac{1}{\hbar}\sqrt{M(V_0 - \epsilon)}$$

$$r > a \quad u = Be^{\pm \alpha r} \qquad : \alpha = \frac{1}{\hbar}\sqrt{M\epsilon}$$

Boundary conditions $u = 0$ $r = 0$ & ∞
u & $\frac{du}{dr}$ continuous at $r = a$

$$u = A \sin Kr \qquad r < a$$
$$u = A \sin Ka\, e^{-\alpha(r-a)} \qquad r > a$$

$$\underline{K \cot Ka = -\alpha}$$

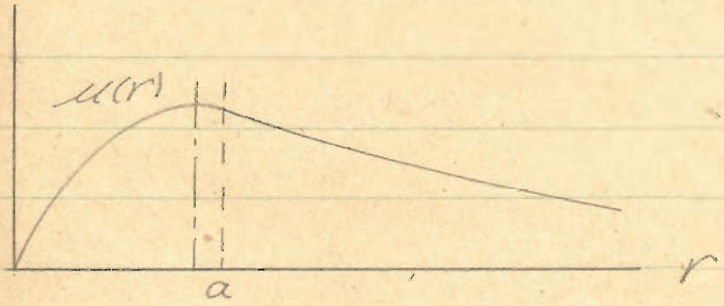
This last formula provides us

with a relation between the depth and width of the potential well.

We have given qualitative arguments to show $V_0 >> \epsilon$. This means $K >> \alpha$

$$\cot Ka = -\frac{\alpha}{K} = -\delta$$

$$Ka = \frac{\pi}{2} + \delta$$



Quite in general, the wave function $u(r)$ will have the form indicated above for the ground state. If the potential goes to zero at $r = a$, $d \ln u / dr = -\alpha$.

Since δ small:

$$Ka \approx \pi/2$$

$$\frac{M V_0 a^2}{\hbar^2} \geq \frac{\pi^2}{4}$$

$\frac{\hbar^2}{M} \sim .40$ Mev-barns

$V_0 a^2 \sim 1$ Mev-barn

For $a = .28 \times 10^{-12}$ cm, $V_0 = 12.5$ Mev.
$a = 10^{-13}$ cm, $V_0 = 100$ Mev.

Problem: Compute $V_0 a^2$ in Mev-barns and V_0 in Mev for $a = .5, 1, 2, 2.8 \times 10^{-13}$ cm.

The potential $V = V_e e^{-\frac{r}{b}}$ is treated in RMP Part A. Include values of $V_e b^2$ in problem. $V_0 a^2$ increases with a. Take

$$\frac{V_0 a^2}{V_0 a^2 (a=0)} = f_0(a)$$

and

$$\frac{V_e b^2}{V_e b^2 (b=0)} = f_e(b)$$

Plot $f_0(a)$ & $f_0(b)$ choosing scale of a & b so that the two curves coincide as nearly as

possible.

The following potentials have been investigated.

Gaussian

$$V = - V_g e^{-\left(\frac{r}{c}\right)^2}$$

Yakawa

$$V = - \frac{g^2}{r} e^{-r/d}$$

Ref. Coppert, Hair & Sachs PR 1939

Double square well.

Ref. Bohm & Richman (unpublished)

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I Normalization

For a square well:

$$V_0 a^2 \gtrsim \frac{\pi^2}{4} \frac{\hbar^2}{M}$$

Let us investigate the normalization of the wave function

$$\int |\Psi(r)|^2 dr = 1$$

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$$\psi(r) = \frac{u(r)}{r}\frac{1}{\sqrt{4\pi}}$$

$$u = Be^{-\alpha r} \qquad r > a$$

$$u = A \sin Kr \qquad r < a$$

where

$$\alpha = \frac{\sqrt{M\epsilon}}{\hbar} \qquad K = \frac{\sqrt{M(V-\epsilon)}}{\hbar}$$

$$\int u^2 dr = 1$$

$$B^2 \int_a^\infty e^{-2\alpha r} dr = \frac{1}{2\alpha} B^2 e^{-2\alpha a}$$

For $\alpha a \ll 1$ which is equiv. to $\epsilon \ll V_0$ $\quad 1 \approx B^2/2\alpha$

$$B \approx 1/\sqrt{2\alpha}$$

$$u \approx \sqrt{2\alpha}\, e^{-\alpha r}$$

This approx. is good at large distances.

$$r < a \qquad u = \frac{Be^{-\alpha a}}{\sin Ka} \sin Kr$$

$$\int_0^a \sin^2 Kr \approx \frac{1}{2} a \qquad (r = a \text{ asymp } \tfrac{\pi}{2} + \delta)$$

$$B^2 \int_a^\infty e^{-2\alpha r}\, dr = B^2\left(\frac{1}{2\alpha} - a\right)$$

$$\frac{B^2 e^{-2\alpha a}}{\sin Ka} \int_0^a \sin^2 Kr\, dr = B^2 \frac{1}{2} a$$

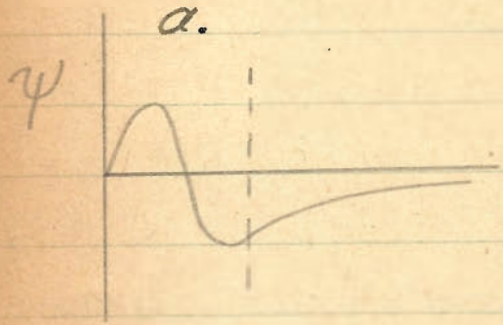
$$B = \sqrt{2\alpha} / \sqrt{1-\alpha a}$$

It is slightly more accurate to write this as

$$B \sim \sqrt{2\alpha}\sqrt{1+\alpha a}$$

II Investigation of Possible Excited States of the Deuteron.

a. $L = 0$



$$K = \frac{\sqrt{M(V_0 - W)}}{\hbar} \sim \frac{\sqrt{MV_0}}{\hbar}$$

$$\cot Ka = -\frac{\alpha}{K}$$

$$Ka \gtrsim 3\pi/2$$

$$V_0 a^2 \gtrsim \frac{9}{4}\pi^2 \frac{\hbar^2}{M}$$

We know from the BE of the

deuteron, H^3 and He^4 that $V_0 \gg \epsilon$. We concluded that $V_0 a^2$ was only slightly greater than $\frac{1}{4}\pi^2 \frac{\hbar^2}{M}$

$\therefore V_0 a^2 < \frac{9}{4}\pi^2 \hbar^2/M$ and there is no excited state with $L = 0$

6. $L = 1$ $\quad \frac{d^2u}{dr^2} + \underbrace{(k^2 - U)}_{K^2} u - \frac{2}{r^2}u = 0$

$r < a \quad \frac{\sin Kr}{Kr} - \cos Kr = u$

$r > a \quad \left(-\frac{1}{\gamma r} + 1\right) e^{-\gamma r} = u \; : \; \gamma = \frac{\sqrt{M}}{\hbar}\sqrt{\omega}$

$r > a \quad \left|\frac{d \ln u}{dr}\right|$ will be least for $r \to a$. Let $e^{-\gamma a} \approx 1$,

$u \approx 1/\gamma r$

$$\left(\frac{d \ln u}{dr}\right)_{a+} < -\frac{1}{a}$$

$$\left(\frac{d \ln (ur)}{dr}\right)_a < 0$$

$r < a \quad u Kr = \sin Kr - Kr \cos Kr$

$$\frac{d(uKr)}{d(Kr)} = \cancel{\cos Kr} - \cancel{\cos Kr} + Kr \sin Kr$$

Hence $\sin Ka < 0$. The first time this occurs is for $Ka > \pi$

$$V_0 a^2 > \pi^2 \frac{\hbar^2}{M}$$

But we know $V_0 a^2 < \frac{\pi^2 \hbar^2}{M}$

∴ no excited state with $L = 1$

Problem: What are V_0 and a if there is just a bound state of the deuteron for $L = 1$ (Square well)

III Scattering by Neutrons and Protons

θ (center of mass) $= 2\chi$ (laboratory)

for particles of equal mass

The scattering x-section is

$$\sigma(\theta) = \left(\frac{\lambda}{2}\right)^2 \left| \sum_l (2l+1)(e^{2i\delta_l} - 1) P_l(\cos\theta) \right|^2$$

Classically $l\lambda \sim b$: $l\hbar = bp$

The l's will not contribute much if $\lambda > a$

In the center of mass system

$$\lambda = \frac{\hbar}{\frac{1}{2}Mv} = \hbar / \sqrt{\tfrac{1}{2}ME}_{lab.}$$

$$= \sqrt{80\,\text{Mev} / E_{lab}} \times 10^{-13}\ \text{cm}$$

At $E = 10$ Mev corresponds to a proton of $\lambda = 2.8 \times 10^{-13}$ cm $\sim a$

Therefore for energies below 10 Mev δ_l will be small, and $\sigma(\theta)$ will be independent of θ.

The scattering in the c.g. system will be isotropic. In the lab. system

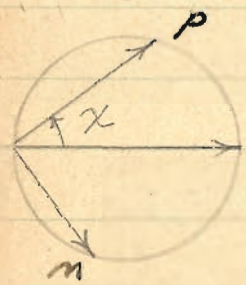
$$\sigma(\theta)\,d\omega \sim \sin 2\chi\, d2\chi$$
$$\sim 4\cos\chi\,[\sin\chi\, d\chi]$$

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I Scattering of protons by neutrons.

In the laboratory system the probability of scattering thru χ is:

$$\cos\chi \sin\chi \, d\chi$$



$$E_P = E\cos^2\chi$$
$$E_N = E\sin^2\chi$$

$$dE_P = 2E\cos\chi\sin\chi\, d\chi$$

~ Laboratory scattering

Calculation of δ_0 (Wigner 1933 Zeits f. Phys.)

$E>0$ $\quad \frac{d^2u}{dr^2} + \frac{M}{\hbar}(E-V)u = 0$

$r>a \quad \sin(kr-\delta_0) \quad : k = \frac{\sqrt{M(E-V_0)}}{\hbar}$

$r<a \quad \sin(Kr) \quad : K = \frac{\sqrt{M(E+V_0)}}{\hbar}$

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Continuity of ln derivative at $r = a$

$$K \cot Ka = k \cot(ka + \delta_0)$$

Let us see what we can find out about δ_0 without involving the potential

$$u \;\|\; \frac{d^2 u_0}{dr^2} + \frac{M}{\hbar}(-\epsilon - V) u_0 = 0$$
$$u_0 \;\|\; \frac{d^2 u}{dr^2} + \frac{M}{\hbar}(E - V) u = 0$$

$$u u_0'' - u_0 u'' - \frac{M}{\hbar^2}(\epsilon + E) u_0 u = 0$$

$$\left[u u_0' - u_0 u' \right]_0^a = \frac{M}{\hbar^2}(\epsilon + E) \int_0^a u_0 u \, dr$$

In general, when $|V| \neq 0$ at $r = a$

$$u_0 u' \;\|\; \left(\frac{d u_0}{dr} / u_0 \right)_a = -\alpha = -\sqrt{M\epsilon}/\hbar$$

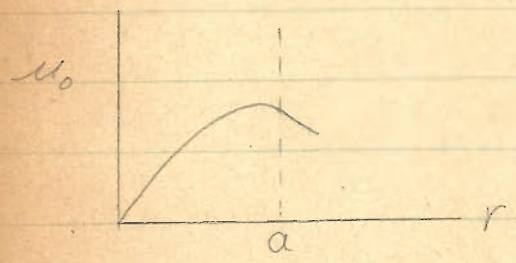
$$u_0 u' \;\|\; \left(\frac{du}{dr} / u \right)_a = k \cot(ka + \delta_0)$$

$$r = 0 \quad u = u_0 = 0$$

$$u u_0 \left(\frac{d \ln u_0}{dr} - \frac{d \ln u}{dr} \right)_a \qquad (A)$$

$$= -\alpha - k \cot(ka + \delta_0)$$

$$= \underbrace{\frac{M}{\hbar^2}(E+\epsilon)}_{\alpha^2 + k^2} \underbrace{\int_0^a u u_0 \, dr \Big/ u_0 u(a)}_{a\gamma}$$



We know

$$u_0(r)_{r<a} \leq u_0(a)$$

If we assume this is also true for $u(r)$, then

$$0 < \gamma < 1$$

$$k \cot(ka + \delta_0) = -\alpha - (\alpha^2 + k^2) a \gamma$$

$$(\alpha a)^2 = \frac{M}{\hbar^2} a^2 \epsilon = \epsilon / V_0 <<< 1$$

$$\alpha a << 1$$

$$E \leq \epsilon = 2.2 \text{ Mev} \quad : \quad k^2 \leq \alpha^2$$

In first approximation: Scattering X-section.

$$k \cot(ka + \delta_0) = -\alpha$$

$$\cot(ka + \delta_0) = -\alpha/k$$

$$\cot \delta_0 = -\alpha/k \qquad (A')$$

$$\sigma_T = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\left(1/(1+\cot^2\delta_0)\right)$$

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$$= \frac{4\pi}{\alpha^2+k^2} = \frac{4\pi k^2}{M}\frac{1}{E+\epsilon}$$

E = energy in c.m. system = $\frac{1}{2}$ energy in lab system.

Second approximation

$$\cot(ka+\delta_0) = -\frac{\alpha}{k} - \frac{\gamma(\alpha^2+k^2)a}{k}$$

$$\cot\delta_0 - \frac{ka}{\sin^2\delta_0} = \cot\delta_0 - ka\left(1+\frac{\alpha^2}{k^2}\right)$$

$$= -\frac{\alpha}{k} - \frac{\gamma(\alpha^2+k^2)a}{k}$$

$$\cot\delta_0 = -\frac{\alpha}{k} + \frac{a}{k}(\alpha^2+k^2)(1-\gamma)$$

$$\sin^2\delta_0 = 1/\left(1+\frac{\alpha^2}{k^2} - \frac{2\alpha a}{k}(\alpha^2+k^2)(1-\gamma) + O(a^2)\right)$$

$$= 1/\left(1+\frac{\alpha^2}{k^2}\right)(1-2\alpha a(1-\gamma))$$

$$\sigma = \sigma_0 (1 + 2\alpha a(1-\delta)) + O(\alpha^2 a^2)$$

To this approximation δ is independent of energy, hence the cross-section is independent of energy. To show δ independent of E, we note in examining equations A and A' that

$$(d \ln u / dr)_a \sim (d \ln u_0 / dr)_a$$

$$\therefore \quad u \sim u_0$$

$$\delta \propto \tfrac{1}{a} \int_0^a u_0^2 \, dr / u_0^2(a)$$

Which is independent of E

At this point let us assume a square well. $V = -V_0 \quad r < a$

$$V = 0 \quad r > a$$

$$u_0 \sim \sin Kr$$

$$\delta \approx \tfrac{1}{a} B^2 \int_0^a \sin^2 Kr / B^2$$

$$\delta \approx \tfrac{1}{2}$$

We now investigate how δ is related to the normalization of

$u_0(r)$ for an arbitrary well.

$$u_0 = Be^{-\alpha r} \qquad r > a$$
$$u_0 = u_0(r) \qquad r < a$$

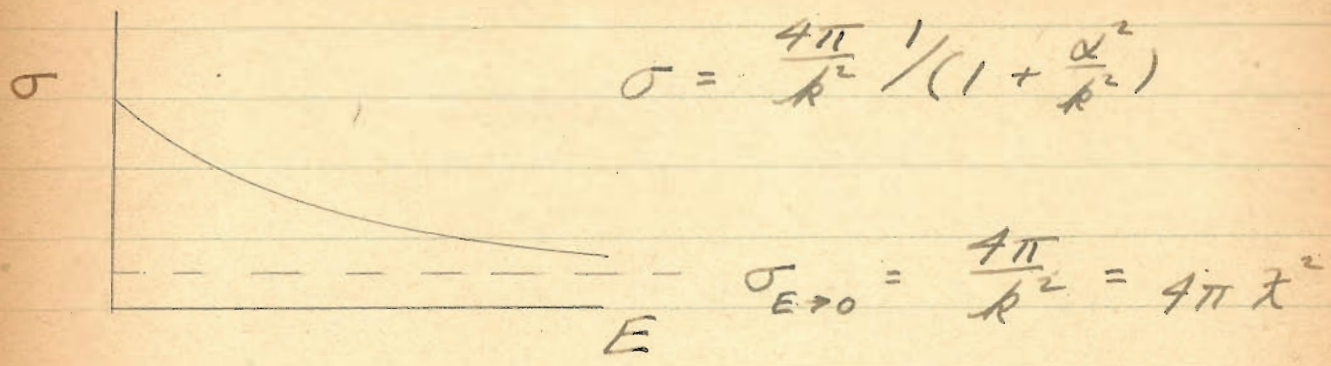
$$\int_a^\infty B^2 e^{-2\alpha r} = \frac{B^2}{2\alpha}(1 - 2\alpha a)$$

$$\int_0^a u_0^2 dr \approx \frac{B^2}{u_0^2(a)} \int_0^a u_0^2 dr = B^2 a \gamma$$

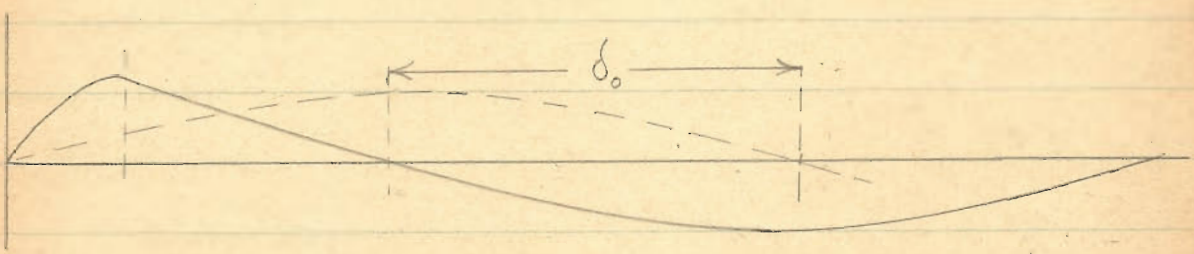
$$(B^2/2\alpha)(1 - 2\alpha a + 2\alpha a \gamma) = 1$$

$$B^2 = 2\alpha(1 + 2\alpha a(1 - \gamma))$$

The first order normalization was $B = \sqrt{2\alpha}$. We see that the correction to the first order normalization is the same as the correction to the first order scattering. This is important in considering photo disintegration.



$4\pi\lambda^2$ is the maximum σ obtainable if only $\delta_0 \neq 0$. To get the maximum $\sin \delta_0 \sim 1$, $\delta_0 \sim \pi/2$. From $\cot \delta_0 = -\alpha/k$, we see that as $k \to 0$ $\cot \delta_0 \to 0$ and $\delta_0 \sim \pi/2$. This is true as long as k is not too large for the approximation we have made to hold. For extremely large k, $\delta_0 \to 0$. For small k $\delta_0 \to \pi$



For small k, the phase shift becomes approximately π.

For $E \to 0$

$$\sigma_0 = 4\pi \cdot \frac{.40 \text{ Mev-barns}}{\epsilon = 2.2 \text{ Mev}}$$

$$\sim 2.4 \text{ barns}$$

The observed σ is about 20 barns. It is interesting to note how σ has changed thru the years. It was first measured by Fermi as 30, it was corrected to account for the chemical binding energy which is important at low energy by a factor 3.3 to 9 barns, it was remeasured as 50 ± 15, it was again remeasured as 55 and the correction changed to 2.7 yielding 20.

The second order approx. as well as the exact treatment give a maximum $\sigma \sim 4$ barns.

The explanation of the discrepancy was given by Wigner. Wigner postulated spin dependent forces between proton and neutron, hence it was incorrect for us to assume the potential for the bound and unbound states the same. We can use the observed σ to calculate V for $S=0$

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We shall derive the n-p potential for $S=0$ from the known connection and the $S=1$ potential

From the deuteron binding, we found

$$\sigma_0 = 4\pi \sin^2\delta_0 / k^2 = \frac{4\pi}{\alpha^2+k^2}$$

Let the logarithmic derivative at the edge of the nucleus be β for $S=0$. Then $\sigma_0 = 4\pi/\beta^2+k^2$.

V8

Since $S=1$ is a triplet state and $S=0$ is a singlet, the x-section for neutrons with random spin is:

$$\sigma = \pi\left(\frac{3}{\alpha^2+k^2} + \frac{1}{\beta^2+k^2}\right)$$

$$= \pi\frac{\hbar^2}{M}\left(\frac{3}{\epsilon_1+E} + \frac{1}{\epsilon_0+E}\right)$$

where E is the energy in the c.m system. If the corrections $1+\alpha a$ and $1-\beta a$ are made, we get $\epsilon_0 = 64$ Kev $<<$ $\epsilon_1 = 2.18$ Mev. To get this result we assume the same range of forces for the singlet and triplet states, since it leads to only a small correction.

The spin functions are

$$S=1 \begin{cases} M=1 & \alpha(N)\alpha(p) \\ M=0 & \frac{1}{\sqrt{2}}\left[\alpha(N)\beta(p) + \beta(N)\alpha(p)\right] \\ M=-1 & \beta(N)\beta(p) \end{cases}$$

$$S=0 \qquad M=0 \qquad \frac{1}{\sqrt{2}}\left[\alpha(N)\beta(p) - \beta(N)\alpha(p)\right]$$

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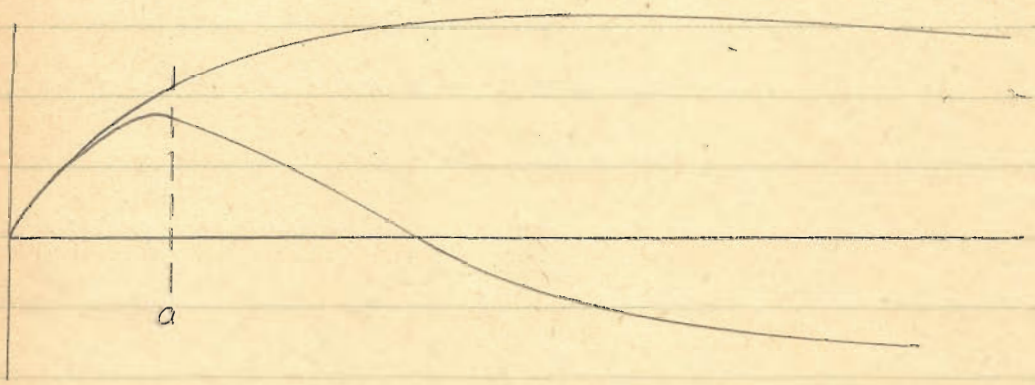
There is no a priori way of knowing whether β is positive or negative.

$$k \cot(ka + \delta_0) = \beta$$

$$\cot \delta_0 = \beta / k$$

$$\sin^2 \delta_0 = 1 / 1 + (\beta/k)^2$$

From this we see that β is an even function of β. We can inquire into the geometrical significance of the sign of β.



We can see that if $\beta > 0$, $\delta_0 = 0$ for $k \to 0$ and if $\beta < 0$ $\delta_0 = \pi$ for $k \to 0$. The important thing is that if $\beta > 0$ there can be no singlet state, since decreasing energy decreases the curvature,

while $\beta < 0$ forces the existance of the singlet state with a binding energy of about 64 kev.

Thus we know that β is positive, and we have what is called a virtual singlet state, with the same resonance effect on the cross-sections. We can see now that the correction factor $1 - \beta a$ makes no real sense, since it is a virtual state.

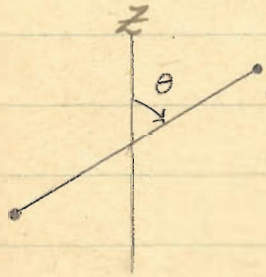
To get more direct evidence on the n-p force, Teller suggested scattering neutrons by ortho and para hydrogen.

Since protons obey Fermi statistics (since the Pauli exclusion principle), the wave function must be antisymmetric to the exchange of two protons.

$\Psi = \Psi(\text{electron})\,\Psi(\text{vibration})$

$\cdot\, \Psi(\text{rotation})\,\Psi(\text{nuclear spin})$

Ψ(vibration) must be symmetric since it depends only on the distance between the protons. Ψ(electron) turns out to be symmetric in the ground state of the H_2 molecule. Ψ(n. spin) is symmetric in the triplet state, and antisymmetric in the singlet state, so Ψ(rotation) must be antisymmetric in the triplet state and symmetric in the singlet state.



$$\Psi_{rot} = P_j^m(\cos\theta)\, e^{im\phi}$$

If we interchange the particles, $\phi \to \pi + \phi$, $\theta \to \pi - \theta$

Then $e^{im\phi} \to (-1)^m e^{im\phi}$;

$P_j^m \to (-1)^{j-m} P_j^m$; $\Psi \to (-1)^j \Psi$.

This is symmetric for even j and antisymmetric for odd j. For ortho hydrogen j is odd (1) and for para hydrogen j is even (0). The nuclear moment is very hard to turn around because it is so weak, so one form of hydrogen takes a long time to degenerate into the other. Parahydrogen is stable at low temperatures but it is possible to preserve ortho hydrogen because of conversion difficulties. To get pure parahydrogen, it is necessary to use a catalyst like O or a paramagnetic gas. The natural ratio is 3 to 1.

Scattering

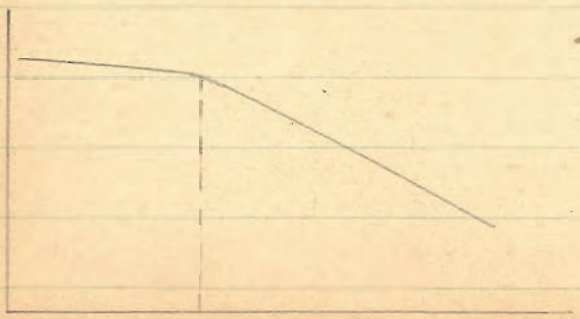
1/14/49

$$\sigma = \frac{3}{4}\frac{4\pi}{\alpha^2 + k^2}(1+\alpha a) + \frac{1}{4}\frac{4\pi}{\beta^2 + k^2}$$

The spin 0 state is a virtual state, which means the $\ln$ derivative is small. It is convenient to use the correction factor $(1-\beta a)$ which comes from same considerations as with real quantum states. This gives a better fit over a larger range of energies.

$$\sigma = \frac{\pi \lambda^2}{M}\left[\frac{3(1+\alpha a)}{\epsilon + E} + \frac{1-\beta a}{\epsilon' + E}\right]$$

where $\epsilon' = \frac{k^2\beta^2}{M}$ and $E = \frac{1}{2}E_L$ (lab system)



The exp. curve gives
$\epsilon' = 64$ Kev

Ref. Frisch
PR 70, 591

Suppose we assume the neutron spin is not 1/2. What is the result.

From the spin of the deuteron it must be 3/2. Then the two states of the deuteron are 1 or 2 and the statistical weights would be 3 and 5 rather than 1 and 3.

$$\sigma = \frac{4\pi\hbar^2}{M}\left[\frac{3}{8}\frac{(1+\alpha a)}{(\epsilon+E)} + \frac{5}{8}\frac{(1-\beta' a)}{(\epsilon''+E)}\right]$$

Problem: Assume $\sigma = 21$ barns for $E = 0$ and calculate σ at various energies. Compare with Fuchs's curve (PR 70 591)

I Low Energies

Some effects at very low energies which enter into scattering:

1. Chemical binding
2. Molecular velocities
3. Spin (ortho & parahydrogen)

If the proton is bound the scattering is no longer at right angles. The binding makes the proton appear heavier. Then isotropic scattering in the c. m. system is also isotropic in the lab. system. The total x-section is also effected. It is increased by a factor 4.

One direct way to get the scattering x-section is to multiply $|V_{on}|^2$ by dN/dE. The system can be normalized in a box and the scattering characterized by θ. The differential x-section is proportional to $d\omega\, p^2 dp/dE$. Since the molecule is heavy, E is the neutron energy. If the molecule

is just a proton, then the energy of the recoil proton must be added. For $\theta = 0$ there should be no difference between scattering by bound & unbound protons since the recoil is at right angles.

There is another way to get the same result. This is very useful but not too clear. We can try to use the Born approximation. It is difficult to see why it should work since it is valid only when the energy of interaction is small compared to the KE of the incident particle. In our case, the interaction = 0 (20 Mev) and the KE $\doteq$ 0 ($\frac{1}{40}$ v). Fermi avoided this difficulty by using a fictitious potential which gave the same scattering, but fulfilled

the conditions for the Born approx

For isotropic scattering

$$\psi = e^{ikz} + \frac{a}{r} e^{ikr}$$

$|a|^2 d\omega$ is the differential x-section.

Suppose a batch of nuclei is a molecule. We can use the theory on each nucleus, provided a plane incident wave is correct for the particles falling on each nucleus. This condition is fulfilled because the distance between nuclei is large compared to the $\sqrt{\sigma} = a$ and hence $a/r \ll 1$ at any other nucleus. Thus each nucleus sends out a scattered wave and they interfere.

Any interaction potential which gives the right value of "a" is all right. If the range

of the forces is R', $\sigma \sim V R'^3$
Thus if we spread the potential over a radius 5×10^{-10} instead of 10^{-12}, this reduces the potential to the region of the Born approx. R' must be small compared to the range of oscillation of the protons in the molecule. It is.

We can now use the approx. which says.

$$\sigma d\omega = \frac{d\omega \mu^2}{(2\pi\hbar^2)^2} \left| \int V(|\underline{R}-\underline{r}|) e^{i\underline{q}\cdot\underline{R}} \psi_0(\underline{r}) \psi_n(\underline{r}) d\underline{R} d\underline{r} \right|^2$$

where $\underline{q} = (\underline{p}_0 - \underline{p})/\hbar$. The proton should be considered as an oscillator since it is bound. For an H-molecule take the right wave function.

Proton in heavy molecule $\mu = M$

Free proton $\mu = \frac{1}{2} M$

This already gives a factor

4, if the V_{nm} does not change.
For free protons

$$\psi_0(r)\psi_n(r) = e^{-i\underline{q}\cdot\underline{r}}$$

Then $e^{i\underline{q}\cdot[\underline{R}-\underline{r}]} \approx 1$ on the average since V varies more slowly.

For a bound proton, consider q small compared to 1/(amp. of osc. of proton in the molecule) This implies $i\underline{q}\cdot\underline{r} << 1$ and $e^{i\underline{q}\cdot\underline{r}} = 1$. The integral becomes $\int V d\tau$, since V is large only when $\underline{R} \approx \underline{r}$ and the ψ_n are orthogonal. So just the factor 4 is left. Same as momentum considerations, but this is more useful because it applies to q not so small for appreciable θ.

The factor 4 appears only for low energies. For higher energies the factor decreases. The factor is a little greater than 1

when the neutron energy is kv
of the proton in the molecule.

Ref Part B pp 122-127

4/18/47

I Recapitulation

In order to be able to apply the Born approximation to the problem of neutrons scattered by bound protons, we invented a fictitious V having a longer range but a smaller value value than the real V_0. This V has the property that it gives the correct cross-section. The range of V, while larger than the true range, must be such that $qx << 1$. Then

$$f(\theta) = \frac{2M}{4\pi\hbar^2} \int V(|\underline{R}-\underline{r}|) e^{i\underline{q}\cdot\underline{R}} \psi_0(\underline{r}) \psi_m(\underline{r}) d\underline{R} d\underline{r}$$

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where $\underline{R}$ is the space coordinate and $\underline{r}$ is the coordinate of the proton scatterer. We can perform the $\underline{R}$ integration first:

$$-\frac{2\mu}{4\pi\hbar^2}\int\psi_0\psi_n^*\,d\underline{r}\int V(|\underline{R}-\underline{r}|)\,e^{i\underline{q}\cdot\underline{R}}\,d\underline{R}$$

$$\cdots\; e^{i\underline{q}\cdot\underline{r}}\int V(|\underline{R}-\underline{r}|)\,e^{i\underline{q}\cdot(\underline{R}-\underline{r})}\,d(\underline{R}-\underline{r})$$

The scattering by a free proton is given by:

$$f(\theta)_{free} = -\frac{2\mu_{free}}{4\pi\hbar^2}\int V(|\underline{R}-\underline{r}|)\,e^{i\underline{q}\cdot(\underline{R}-\underline{r})}\,d(\underline{R}-\underline{r})$$
$$= B$$

Substituting in the bound scattering formula, we obtain:

$$f(\theta) = \frac{\mu}{\mu_{free}}B\int e^{i\underline{q}\cdot\underline{r}}\psi_0(\underline{r})\psi_n^*(\underline{r})\,d\underline{r}$$

Since V is large only for $q|\underline{R}-\underline{r}| \ll 1$, the expression

$\int d(\underline{R}-\underline{r})\, e^{i\underline{q}\cdot(\underline{R}-\underline{r})} V(|\underline{R}-\underline{r}|)$ is simply

$\int V(x)\, d\tau$ since $e^{i\underline{q}\cdot(\underline{R}-\underline{r})} \approx 1$

II Systems of Several Nuclei

Let:

$\underline{r}_k$ be the coordinate of the k^{th} nuclii and b_k the free scattered amplitude from the k^{th} nuclii. Then

$$f(\theta) = \mu \int \Psi_0 \Psi_n^* (\ldots r_k \ldots) \sum_k \frac{b_k}{\mu_k} e^{i\underline{q}\cdot\underline{r}_k} \ldots d\tau_k \ldots$$

where: $\mu_k = \dfrac{M M_k}{M + M_k}$

$$\mu = \frac{M \sum_k M}{M + \sum_k M}$$

Consider an H_2 molecule

$$\mu = \frac{M\,2M}{M+2M} = \frac{2}{3} M$$

$$\mu_k = \frac{1}{2} M$$

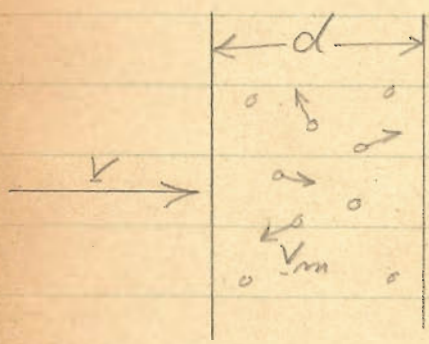
Ref. on scattering R.M.P.

In general, for any energy the maximum x-section is approx

$$< 1 + \frac{\hbar\omega}{E}$$

III The moving molecule effect.

In a gas the molecules have random thermal motion, $\underline{V}_m$. The x-section for an individual molecule would be $\sigma(|\underline{V} - \underline{V}_m|)$.



Suppose a particle traverses a slab of thickness d. The number of collisions will be:

$$\# = N \left\langle \sigma(|V - V_m|)(|\underline{V} - \underline{V}_m|) \right\rangle_{av} \left(\frac{d}{V}\right)$$

The x-section will be

$$\sigma_{eff} = \frac{1}{V} \left\langle \sigma(|\underline{V} - \underline{V}_m|)(|\underline{V} - \underline{V}_m|) \right\rangle_{av}$$

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If $V_m << V$ $\sigma_{eff} = \sigma(V)$

If $V_m \gg V$ $\quad \sigma_{eff} \sim \frac{1}{V}$

For low energy $\sigma_{eff} = \sigma \frac{\langle |\underline{V} - \underline{V}_m| \rangle_{av}}{V}$

IV Effect of Spin

Wave function for a diatomic molecule:

$$\psi_{el} \quad \psi_{vib} \quad \psi_{rot} \quad \psi_{nuclear\ spin}$$

ψ_{el} mostly symm
ψ_{vib} symm
ψ_{rot} { symm. j even
anti j odd
$\psi_{n.s.}$ { symm $s_t = 1$ (j odd)
anti $s_t = 0$ (j even)

Ortho H_2 weight 3 spin 1
Para H_2 " 1 " 0 Stable at low temperature.

Let:

a_1 = scattered amplitude if resultant spin of $n \& p = 1$

a_0 = scattered amplitude if resultant spin of $n \& p = 0$

We were able to deduce a_1 from the deuteron potential. Measurements of the scattering x-section then gave us a_0

$$\sigma_{free} = 4\pi\left(\frac{1}{4}a_0^2 + \frac{3}{4}a_1^2\right) = 21 \text{ barns}$$

$$a_0^2 + 3a_1^2 = 6.7$$

$5.7 \qquad 1$

Scattered Amplitude: Pauli Spin Operators.

$$A_{np} = A + B\,\underline{\sigma}_N \cdot \underline{\sigma}_P$$

where $\underline{\sigma}$ = Pauli Spin operators. (For spin $\frac{1}{2}$)

$\sigma_x \alpha = \beta \qquad \sigma_y \alpha = -i\beta \qquad \sigma_z \alpha = \alpha$

$\sigma_x \beta = \alpha \qquad \sigma_y \beta = i\alpha \qquad \sigma_z \beta = -\beta$

$$\sigma_x^2 \alpha = \alpha, \quad \sigma_y^2 \alpha = \alpha, \quad \sigma_z^2 \alpha = \alpha$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

$$\sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$$

$$\sigma_y \sigma_z = -\sigma_z \sigma_y = i\sigma_x$$

$$\sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y$$

$\underline{\sigma}$ = spin angular momentum in units of $\frac{1}{2}\hbar$

$\underline{S}$ = spin angular momentum in units of $\hbar$

$$\underline{S} = \frac{1}{2}\underline{\sigma}$$

$$\underline{\sigma}_N \cdot \underline{\sigma}_P = \sigma_{Nx}\sigma_{Px} + \sigma_{Ny}\sigma_{Py} + \sigma_{Nz}\sigma_{Pz}$$

$$\underline{S} = \underline{S}_N + \underline{S}_P = \frac{1}{2}(\underline{\sigma}_N + \underline{\sigma}_P)$$

$$S^2 = \frac{1}{4}(\sigma_N^2 + \sigma_P^2) + \frac{1}{2}\underline{\sigma}_N \cdot \underline{\sigma}_P = S'(S'+1)$$

$$\underline{\sigma}_N \cdot \underline{\sigma}_P = 2S'(S'+1) - \frac{1}{2}(3+3)$$
$$= 2S'(S'+1) - 3$$

			Weight
n-p spin	$S' = 1$	$\underline{\sigma}_N \cdot \underline{\sigma}_p = 1$	3
n-p spin	$S' = 0$	$\underline{\sigma}_N \cdot \underline{\sigma}_p = -3$	1

The average with statistical weights equals zero.

$$a_0 = A - 3B$$
$$a_1 = A + B$$

$$A = \frac{a_0 + 3a_1}{4}$$
$$-B = \frac{a_0 - a_1}{4}$$

We know the magnitude & sign of a_1 from study of deuterium. We know the magnitude of a_0 from measured σ and a_1.

A measurment of a_1 would enable one to deduce the range of the nuclear forces, α.

$$a_1 = -\frac{1}{\alpha}(1 + \alpha a)$$

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I The amplitude scattered from a free proton is:

$$A + B \underline{\sigma}_N \cdot \underline{\sigma}_P$$

If the resultant spin of n and p

a) $S = 0$ $\quad a_0 = A - 3B$

b) $S = 1$ $\quad a_1 = A + B$

Let us consider the amplitude of the wave scattered from the two protons in a H_2 molecule.

$$a = (A + B \underline{\sigma}_N \cdot \underline{\sigma}_{P_1}) e^{i\underline{q} \cdot \underline{r}_1} + (A + B \underline{\sigma}_N \cdot \underline{\sigma}_{P_2}) e^{i\underline{q} \cdot \underline{r}_2}$$

where $\underline{q} = \underline{k}_0 - \underline{k}_n$

Let us use center of mass coordinates. Then $\underline{r}_1 = -\underline{r}_2$

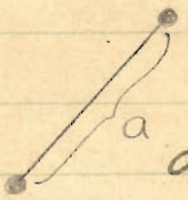
We shall consider neutrons of low energy (20°). Hence our molecule will always be in the ground state except for rotation. The energy levels of the rotational states are

$$E_j = B\,j(j+1) \quad \text{where } B = 57\,\text{cm}^{-1}$$

For 20° neutrons we can get transitions from $j=1$ to $j=0$ but none the other way.

At low energies we have

$$p/\hbar\, a \ll 1$$



$a = .75\,\text{Å}$

$\lambda = \hbar/p = .1\,\text{Å}$ at .2 v

$\lambda = \hbar/p = 1\,\text{Å}$ at .002 v ~ 20°

$$k a = p/\hbar\, a = .75.$$

$$|r_1| = |r_2| = \tfrac{1}{2} a$$

$$\therefore \quad |q \cdot \underline{r}| < .4$$

And $e^{i q \cdot r} \sim 1$

At low energies, thus, the scattered amplitude from two protons is simply the sum of the scattered amplitudes from the individual protons with no phase factor to consider.

$$a = 2A + B\,\underline{\sigma}_N \cdot (\underline{\sigma}_{p_1} + \underline{\sigma}_{p_2})$$

$$\text{but } \underline{\sigma}_{p_1} + \underline{\sigma}_{p_2} = 2(\underline{S}_{p_1} + \underline{S}_{p_2}) = 2\underline{S}_H$$

$$a = 2(A + B\,\underline{\sigma}_N \cdot \underline{S}_H)$$

$$\underline{S}_H \chi_{ortho} = \lambda \chi_{ortho}$$
$$\underline{S}_H \chi_{para} = \lambda \chi_{para}$$

$\underline{S}_H$ is diagonal hence in the approximation $e^{i\underline{q}\cdot\underline{r}} \sim 1$ we get no transitions from ortho to para hydrogen (or vice versa)

$$a_{on} = \int g(r)\, \psi_H^0(r)\, \psi_H^N(r)\, d\tau$$

$$\psi_H^0 = \psi_H^N$$

Hence the scattering is independent of the coordinates of the H_2 molecule.

The probability of scattering is given by $|A|^2$ and is isotropic.

$\frac{1}{4}\left[\,|A|^2\,\right]$ averaged over all $\underline{\sigma}_N \cdot \underline{S}_H$

$$= \left\langle A^2 + 2AB \underbrace{\underline{\sigma}_N \cdot \underline{S}_H}_{0} + B^2 (\underline{\sigma}_N \cdot \underline{S}_H)^2 \right\rangle$$

$$(\underline{\sigma}_N \cdot \underline{S}_H)^2 = (\sigma_{Nx} S_{Hx} + \sigma_{Ny} S_{Hy} + \sigma_{Nz} S_{Hz})^2$$

$$= \left.\begin{aligned} &\sigma_{Nx}^2 S_{Hx}^2 + \sigma_{Ny}^2 S_{Hy}^2 + \sigma_{Nz}^2 S_{Hz}^2 \\ &+ \sigma_{Nx}\sigma_{Ny} S_{Hx} S_{Hy} + \sigma_{Ny}\sigma_{Nx} S_{Ny} S_{Hx} + \dots \end{aligned}\right\} = \underline{S}_H^2 = 0$$

$$\underline{S}_H^2 = S(S+1)$$

$$= 0 \quad \text{Para } H_2$$

$$= 2 \quad \text{Ortho } H_2$$

$$\frac{1}{4}|a|^2 = A^2 + B^2 S(S+1)$$

Para $= A^2$

Ortho $= A^2 + 2B^2$

The scattering by the natural mixture of ortho and para H_2 will be:

$$P + 3O = 4A^2 + 6B^2$$

We can compare this with the scattering from two free protons:

$$2P = 2A^2 + 6B^2$$

We see that we get constructive interference from the H_2 molecule for the A terms which give a factor 2 in amplitude and 4 in intensity. For the spin dependent B terms the intensities only add.

Let us now investigate the

relation of A and B to the scattering amplitudes a_0 and a_1

We obtained a_1 from the deuteron

$$a_1 = -\frac{1}{\alpha} - \frac{1}{2}a$$
$$= -.43 - .5a$$

where a = range of nuclear forces.

We obtained a_0 from the exp. scattering x-section $\sigma = 21$ barns

$$21 = \pi a_0^2 + 3\pi a_1^2$$
$$a_0 = \pm(2.48 - .26a)$$

The sign of a_0 is in doubt.

$$A = \frac{a_0 + 3a_1}{4}$$

$$-B = \frac{a_0 - a_1}{4}$$

Assume first $a_0 > 0$

$$A = \frac{1.20 - 1.76a}{4} = .30 - .44a$$

$$-B = \frac{2.91 + .24a}{4} = .72 + .06a$$

Assuming $a = 2.8 \times 10^{-13}$

$A = .18$
$B = .74$

Para $\sigma_p \sim .030$
Ortho $\sigma_o \sim 1.09$

$\sigma_o : \sigma_p = 35$

Now assume $a_o < 0$

$-A = .94 + .31a$
$B = .51 - .06a$

Para $\sigma \sim 1.06$
Ortho $\sigma \sim 1.56$

$\sigma_o : \sigma_p = 1.5$

Experimental results will determine the sign of a_o. It turns out

that $a_0 > 0$ and hence the singlet state is virtual as we have already postulated.

Problem: Assume the neutron spin is 3/2 and carry out the above calculation. It turns out that with this assumption it is impossible to explain the large ratio between ortho and para scattering which is actually observed.

Experiments on ortho and para scattering were carried out by Alvarez & Pitzer 1939 and Dewire (Z A 1946) Ref: RMP-B The experiments are very difficult to carry out.

The ortho scattering is simply related to the free scattering. The fact that ortho converts

to para H_2 at low temperatures complicates this check on the experiment however.

Ortho x section

E	σ
.0009 v	145 barns
.0024 v	115
	average 120 barns

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I Scattering by Ortho and Para Hydrogen. (cont.)

A) Range of nuclear forces from para scattering

$$\sigma_p \sim A^2$$

$$A^2 = (.30 - .44a) \times 10^{-12} \text{ cm}$$

The experimental x-section for para scattering.

$E = 1.5 - 2.4$ millivolts

$\sigma = 3.9$ barns

The value of A deduced

from this x-section after making corrections for molecular motion, etc. is

$$A = .22 \times 10^{-12} \text{ cm}$$

This gives a value of the range

$$a = .18 \times 10^{-12} \text{ cm.}$$

The value of the range deduced from p-p scattering and n-p scattering at high energy is

$$a = .28 \times 10^{-12} \text{ cm} \pm 15\%$$

The agreement is not good.

Assuming a range of $.28 \times 10^{-12}$ cm. for the forces, we can use the theory to derive the "expected" σ para. It turns out

$$A = .17_5 \times 10^{-12}$$

$$\sigma_{\substack{\text{para} \\ \text{expected}}} = 2.2 \text{ barns}$$

One would expect this value to be outside the experimental error. Ref Schwinger P.R. 1939

We can use our scattering x-section formula to check which result is correct

$$\sigma = \pi \frac{\hbar^2}{M} \left[\frac{3(1+2a)}{(E+\epsilon_1)} + \frac{(1-3a)}{(E+\epsilon_0)} \right]$$

where $E = \frac{1}{2} E$ laboratory. At high energies, $E = \frac{1}{2}\epsilon_1$ ($2E = 2.2$ Mev Lab) the x-section is quite sensitive to a. A change $2.8 \rightarrow 1.8 \times 10^{-13}$ in the range esp. to a decrease in σ of 8%. This result favors 2.8×10^{-13} cm. The difficulty has not been resolved. In any case, the ortho & para scattering is the most sensitive to the range of forces.

19

II Interaction of Radiation with Deuterium. Ref: RMP Part A B the

The interactions of radiation with H_2^2 are small and can be calculated using perturbation theory.

A) Disintegration

The probability of disintegration is given by:

$$\sigma_{\gamma_E} = \frac{\omega p M_N}{\hbar^3 c} D^2$$

where ω = frequency of the γ-ray

p = momentum of the c.m. of the system after disintegration

M_N = neutron mass

D = dipole moment between the two states.

The initial energy $= -\epsilon$

" final " $= \hbar\omega - \epsilon$

100 γ's from ThC' carry 2.62 Mev.

This leads to $\epsilon = 2.18$ Mev and K.E. = .44 Mev.

There are two parts to the interaction, the <u>electric</u> and the <u>magnetic</u>

Photo Electric effect.



$$\underline{D} = \int e \tfrac{1}{2} \underline{r}\, \psi_0 \psi_E^* \, d\tau$$

$$\psi_0 = \sqrt{\frac{\alpha}{2\pi}}\sqrt{1+\alpha a}\;\frac{e^{-\alpha r}}{r}\;\chi_1$$

where χ_1 = spin wave function for the triplet state. The spin remains unchanged in the photo electric transition. The final state will be one with $S=1$, $L=1$ (The initial state was assumed $S=1$, $L=0$)

$$\psi_E = u_1(r)\, P_1(\theta\phi)\, \chi_1$$

For small r, $u_1 \sim r^2$.

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Therefore if the range of our forces is small the wave function will not be much perturbed and we can use the $l=1$ component of a plane wave for ψ_E. This is alright up to $E \sim 10$ Mev.

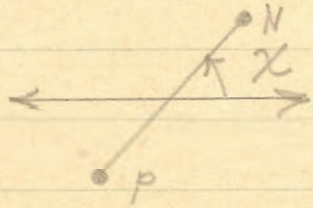
Assume γ's polarized in the x direction. We must take the component of the dipole moment in the x-direction.

$$D_x = \tfrac{1}{2} e \int x \psi_0 \psi_E \, d\tau$$

This yields

$$\sigma_d = 2 d\omega \frac{e^2}{\hbar c} \cos^2 \chi \frac{\alpha k^3}{(\alpha^2 + k^2)^3}$$

where $\alpha = \frac{\sqrt{M\epsilon}}{\hbar}$, $k = \frac{\sqrt{ME}}{\hbar}$

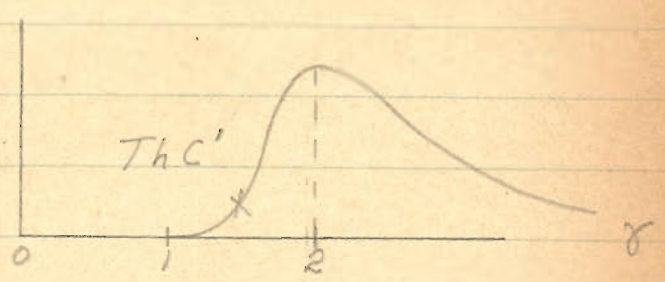


polarization

$$\sigma_d = \frac{8\pi}{3} \frac{e^2}{\hbar c} \frac{\hbar^2}{M} \underbrace{\frac{\sqrt{\epsilon}\, E^{3/2}}{(E+\epsilon_1)^3}}_{\hbar\omega}$$

If we let $\gamma = \frac{h\omega}{\epsilon_1}$

$$\sigma \sim \frac{(\gamma-1)^{3/2}}{\gamma^3}$$



Max σ at 4.4 Mev

$\sigma_M = 1.45 \times 10^{-27}\ cm^2$

The correction factor is $(1+\alpha a)$

Photo Magnetic effect.

The photo magnetic effect is due to the interaction of the δ with the spin of the H and n.

The operator of the magnetic moment is:

$$o\,m\,m = \mu_H \bar{\sigma}_H + \mu_N \bar{\sigma}_N$$

Applying this to the spin wave function

$$(\mu_H \bar{\sigma}_H + \mu_N \bar{\sigma}_N)\chi_1$$

$\mu_H \neq \mu_N$ otherwise we get

$$\mu(\bar{\sigma}_H + \bar{\sigma}_N)\chi_1 = 2\mu S_D \chi_1 = 2\mu s' \chi_1$$

which means no transitions.

$$(\mu_H \underline{\sigma}_H + \mu_N \underline{\sigma}_N) \chi_1 = A\chi_0 + B\chi_1$$

$A \sim (\mu_H - \mu_N)$ ...

We get transitions to $S=0$. L does not change since the o.m.m. is independent of the coordinates. The transitions will be

$$S=1 \rightarrow S=0$$
$$L=0 \rightarrow L=0$$

Also

$$\underline{D} \sim \int \underset{S=1}{\psi_0} \underline{\psi} \underset{S=0}{\psi_E} d\tau$$ which is finite

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Magnetic effect.

The operator associated with the magnetic moment is:

$$\underline{\mu} = \mu_H \underline{\sigma}_H + \mu_N \underline{\sigma}_N$$

The matrix element $\underline{D}$ between the states ψ_0 and ψ_E is

$$\underline{D} = \int \psi_E^* \underline{\mu} \psi_0 \, d\tau$$

This integral breaks up into a sum over the spin coordinates times an integral over the space coordinates:

$$\underline{D} = \sum_\sigma \chi_f^* \mu \chi_i \int \psi_f \psi_i \, d\tau$$

The orthogonality of the radial wave functions requires that the transition be:

$$l = 0 \rightarrow l = 0$$

$$S = 1 \rightarrow S = 0$$

The sum is proportional to $\sigma_H - \sigma_N$. After integration over the radial wave function we have left

$$\int dr \, u_f u_i \quad \text{where } \psi(r) = \frac{u(r)}{r}$$

$$u_i(r) = \sqrt{\frac{2\pi}{a}}\, e^{-ar}\left(1 + \tfrac{1}{2}ra\right)$$

$$u_f(r) = \sin(kr + \delta_{0\,\text{singlet}})$$

The photo-magnetic X-section turns out to be

$$\sigma_{mag} = \frac{2\pi}{3}\,\frac{e^2}{\hbar c}\,\frac{\hbar^2}{M}\,\frac{\sqrt{\epsilon_1}\sqrt{E}\left(\sqrt{\epsilon_1}+\sqrt{\epsilon_0}\right)^2}{(\epsilon_1+E)(\epsilon_0+E)\,Mc^2}\,(\mu_H-\mu_N)^2$$

The term $(\sqrt{\epsilon_1}+\sqrt{\epsilon_0})^2 = (\alpha+\beta)^2 =$ (sum of the log. slopes of the initial and final wave functions at the boundary of the potential well) squared. The effect of this term enables one to deduce the virtual singlet state from measured σ's

The ratio of σ_d to σ_{mag} is

$E \to \infty$

$$\frac{\sigma_{mag}}{\sigma_d} = .013$$

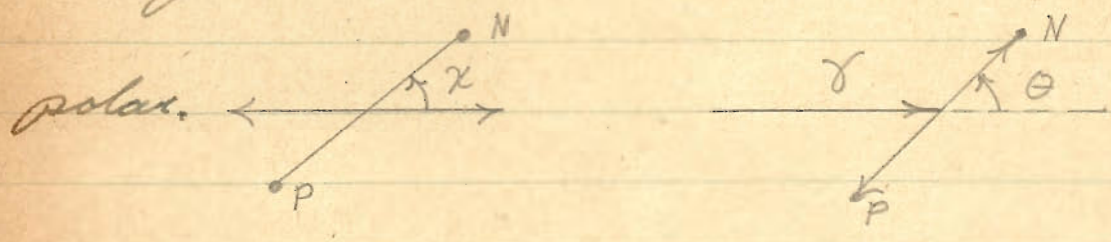
$E \to 0$

$$\frac{\sigma_{mag}}{\sigma_d} = .013\,\frac{\epsilon_1}{E}\,\frac{\epsilon_1}{\epsilon_0+E}$$

$E_{lab} = 2.62$ Mev

$\frac{\sigma_{mag}}{\sigma_d} = .27$

The photo electric and photo magnetic processes have different angular distributions. The photo magnetic disintegration leads to an $L=0$ state which is spherically symmetric. The photo electric effect leads to an $L=1$ state which gives a $\cos^2\chi$ distribution. If we average over all directions of polarization we obtain an angular distribution $\sin^2\Theta$.



The total cross section will be:

$$\sigma_T = A\sin^2\Theta + B$$

$\sigma_{exp} = A \sin^2\theta + .26 \pm .08$

$\sigma_{theo} = A \sin^2\theta + .18$

Ref: Halban Compt Rend 206 1170 1938
(Found no forward scattering)

Graham & Halban Rev Mod Phy
17 297 (1947) (Above result)

Principle of Microscopic Reversibility states that elementary reactions are reversible with equal probabilities

Consider the systems

A = Deuteron + γ

B = neutron + proton.

$$A \rightleftarrows B$$

Suppose we have many such systems in statistical equilibrium at high temperature.

Then:

No. of states of A per unit energy times $\sigma_{A\to B} V_A = N(B)\, \sigma_{B\to A} V_B$

$$N(A) = \frac{4\pi p_A^2 dp_A}{g_A (2\pi\hbar)^3 dE} \quad ; \quad \frac{dE}{dp_A} = V_A$$

g_A is a weight related to the internal degrees of freedom. We obtain

$$\frac{\sigma_{B\to A}}{\sigma_{A\to B}} = \frac{p_A^2 g_A}{p_B^2 g_B}$$

In our case $p_A = \frac{\hbar\omega}{c}$

$$p_B^2 = M_N E_{am.}$$

g_A = 3 (singlet) × 2 (polar)

g_B = 2 (spin) × 2 (spin)

From the above relations and our knowledge of $\sigma_{A\to B}$ we can find the capture x-section, $\sigma_{B\to A}$. The capture x-section is important only at low energies:

$$\sigma_{cap} = \pi \frac{e^2}{Mc^2} \frac{\hbar}{Mc} \frac{\sqrt{E_1}(E_1 + \frac{1}{2}E_0)(\sqrt{E_1} + \sqrt{E_0})^2}{E_0 (E_0 + \frac{1}{2}E_0) Mc^2} (\mu_H - \mu_p)^2$$

E_0 = energy in laboratory.
μ is in nuclear magnetons.

$\sigma_{cap} \sim 1/v$ Therefore the number of capture processes per sec = $\sigma_{cap} v$ which is independent of the energy.

At thermal energies 20°C (.025 ev)
σ_{cap} = .31 barns theoretical
σ_{cap} = .30 barns observed
This is large for a capture x-section and is the reason why water (hydrogen) makes a poor moderator for a chain reactor.

The photo-disintegration of the deuteron gives us no new information about the nuclear forces, but it does show that it is legitimate to assume the proton a point charge when calculating the dipole -

moment.

None of the processes we have discussed gives us any information about the forces in any free state other than for $L = 0$.

If we consider the scattering of neutrons by free protons, we must go to energies of at least 10 Mev before the $L = 1$ term becomes important. For this energy $kR \approx a$. Unfortunately, at this energy the phase shift $\delta_0 \sim \pi/2$ which eliminates the first order $L = 1$ term.

$$f(\theta) = \frac{\lambda}{2i} \sum_l (2l+1)(e^{2i\delta_l} - 1) P_l(\cos\theta)$$

Consider the first two terms $L = 0$ & 1.

$$f(\theta) \sim (e^{2i\delta_0} - 1) + 3(e^{2i\delta_1} - 1)\cos\theta$$

$$\sim \cos 2\delta_0 - 1 + i \sin 2\delta_0 + 6i\delta_1 \cos\theta$$ 111

$$|f(\theta)|^2 \sim (\sin 2\delta_0 + 6\delta_1 \cos\theta)^2 + (RP)^2$$
$$\sin^2 2\delta_0 + 12\delta_1 \sin 2\delta_0 \cos\theta + \ldots$$

We have $\cot(\delta_0 + ka) = -\frac{\alpha}{k} - \frac{(\alpha + k^2)a}{2k}$

For $k \gg \alpha$ $\delta_0 \to \pi/2$ For $\delta_0 \sim \pi/2$ the linear term in δ_1 is very small. For addition

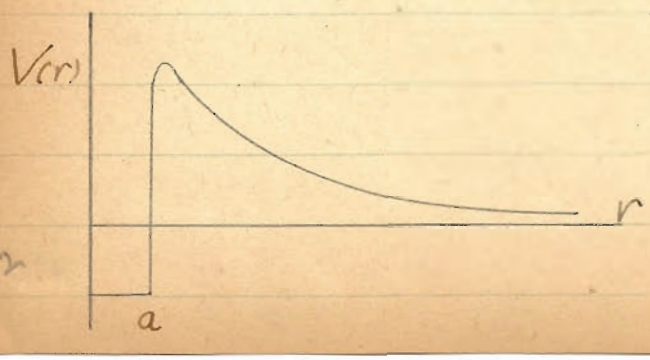
$$\delta_{0\,\text{triplet}} = 90^\circ + \epsilon$$
$$\delta_{0\,\text{singlet}} = 90^\circ - \epsilon$$

which makes for further cancelation. To investigate the $l = 1$ scattering one must go to energies considerably greater than 10 Mev.

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Proton-Proton Scattering

The potential for p-p scattering is as pictured below:



Proton-proton scattering is a combination of the following effects:

1) Coulomb scattering
2) Symmetry effect
3) Nuclear scattering

At low energies, $E < 2.5$ Mev, the only important term in the nuclear scattering is for $l = 0$ The scattering is S wave scattering (spherically symmetric)

1) Coulomb scattering

$$\left[d\sigma = \frac{e^4 Z_1^2 Z_2^2}{4m^2 v^4} \csc^4 \frac{\theta}{2} \, 2\pi \sin\theta \, d\theta \right]$$

The wave function for coulomb field is:

$$\psi = e^{ikz + ia\,k\ln(r-z)} + \frac{g(\theta)}{r} e^{i\phi}$$

where

$$\phi_0 = (kr - a \ln kr + \pi + 2\eta_0)$$

The scattering function is

$$g(\theta) = \frac{e^2}{Mv^2} \frac{1}{\sin^2\frac{\theta}{2}} \exp\left(-id \ln \sin^2\frac{\theta}{2}\right)$$

where $d = \frac{kv}{e^2} = 137\frac{v}{c}$

$$k = Mv/2\hbar$$

The coulomb scattering x-section in the center of mass system corrected for the symmetry effect is:

$$d\sigma = |g(\theta)|^2 d\omega = \frac{e^4}{E_0^2}\left\{\frac{1}{\sin^4\theta} + \frac{1}{\cos^4\theta} - \frac{\cos\left[\frac{e^2}{\hbar v}\ln\tan^2\theta\right]}{\sin^2\theta\cos^2\theta}\right\}\cos\theta\, d\omega$$

The numerator of the last term is often approximated by "1".

2.) Effect of Nuclear Potential

$$V_{coul} = \sum_l V_l$$

where the V_ℓ are the solutions of the wave equation with a coulomb potential with angular momentum $\ell\hbar$. The V_ℓ are assumed to contained the correct amplitude factor.

$$\psi_{nuc} = \sum_\ell u_\ell$$

where u_ℓ is a solution of the wave equation for nuclear + coulomb potential and angular momentum $\ell\hbar$.

We may write ψ_{nuc} as:

$$\psi_{nuc} = \psi_{coul} + \sum_\ell (u_\ell - V_\ell)$$

The difference $(u_\ell - V_\ell)$ will be large (at low energy) only if $\ell = 0$. This follows from the fact that for finite ℓ, the wave function is very nearly zero in the region of the

nuclear potential hole i.e. near the origin.

$$\psi_{nuc} \doteq \psi_{coul} + M_0 - V_0$$

ψ_{nuc} must have the form

$$\psi_{nuc} = e^{ikr + id\,k\ln(r-z)} + \frac{f(\theta)}{r} e^{i\{\ldots\}}$$

where $f(\theta) = g(\theta)$ + a spherically symmetric term.

$$f(\theta) = g(\theta) + \frac{i}{2k}(e^{2i\delta_0} - 1)$$

3) Symmetry effect

The wave function of the entire system must be anti symmetric to the interchange of the two protons.

a) Spin antisym. (singlet state)

∴ Space function symmetric.

$$\psi_{\substack{coul \\ sym}} = \psi(\underline{R}) + \psi(-\underline{R})$$

$$\Psi_{\substack{nuc \\ sym}} = \Psi_{\substack{coul \\ sym}} + 2 \sum_{l\,even} (\mathcal{U}_l - V_l)$$

b) Spin symmetric (triplet state)
∴ Spacial function antisymmetric

$$\Psi_{\substack{coul \\ anti}} = \Psi(\underline{R}) - \Psi(-\underline{R})$$

$$\Psi_{\substack{nuc \\ anti}} = \Psi_{\substack{coul \\ anti}} + 2 \sum_{l\,odd} (\mathcal{U}_l - V_l)$$

As already mentioned the only term in the nuclear scattering is $2(\mathcal{U}_0 - V_0)$

The total x-section will be made up as follows

$$\sigma = \frac{1}{4}\sigma_{singlet} + \frac{3}{4}\sigma_{triplet}$$

The differential x-section including all effects will be:

$$d\sigma = \frac{c}{E_0^2}^4 \cos\theta_1 \, d\omega \left\{ \frac{1}{\sin^4\theta_1} + \frac{1}{\cos^4\theta_1} - \frac{1}{\cos^2\theta_1 \sin^2\theta} \right.$$

$$\left. - \frac{2kV}{c^2} \frac{\sin\delta_0 \cos\delta_0}{\sin^2\theta_1 \cos^2\theta_1} + \left(\frac{2kV}{c^2}\right)^2 \sin^2\delta_0 \right\}$$

The 4th term arises due to interference between coulomb and nuclear scattering.

The 5th term is a spherically symmetric term from the nuclear scattering.

The sign of δ_0 determines whether the potential at small distances is repulsive or attractive. Experiment shows $\delta_0 > 0$ esp to an attractive potential at short distances.

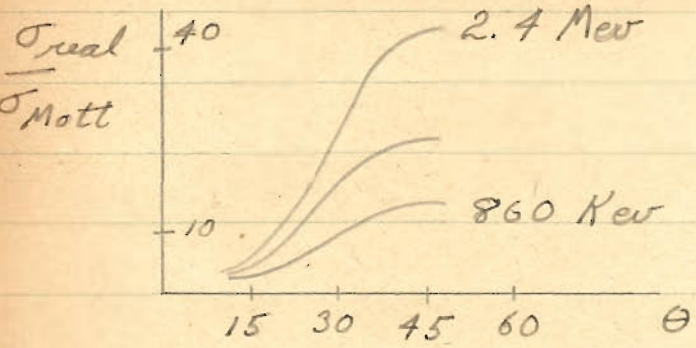
At small angles the coulomb law holds.

At large angles the scattering is greater than the coulomb scattering.

In practice one measures $\frac{\sigma_{real}}{\sigma_{coul}}$

The value of δ_0 is determined by the trend in the x-section as a function of energy.

118 Ref: Herb PR June 1939, 998



The above plot gives an idea of the effect of the nuclear scattering.

It is found that there is no δ_1 effect up to 2.4 Mev.

An analysis of these experiments enables one to determine either the range of forces or the depth of the potential well provided one or the other of these quantities is already known.

e.g. Assuming $a = \frac{e^2}{mc^2}$

Depth = 10.50 ± .05 Mev.

The details of the shape of the potential well are indistinguishable

at low energies, < 2.5 Mev.

5/9/47

I Information from proton-proton scattering:

1) Depth of potential well to 1% (given the width)

2) Range of nuclear forces 2.8×10^{-13} cm $\pm$ 15%

3) Sign of the force - the force between protons is attractive at small distances; there is no stable state since the phase shift goes to zero rather than π at low energy.

4) Force between nuclear particles independent of charge. From p-p scattering $V_0 = 10.5$ Mev
n-p scattering $V_0 = 11.9$ Mev
The difference is just accounted for by the coulomb repulsion

II Non Central Forces.

The quadrapole moment of the deuteron implies non central forces. The only reasonable assumption one can make about these forces is that they depend on the spin as follows

$$S_{12} = 3\frac{(\underline{\sigma}_1 \cdot \underline{r})(\underline{\sigma}_2 \cdot \underline{r})}{r^2} - \underline{\sigma}_1 \cdot \underline{\sigma}_2$$

$$V(r) = V_1(r) + V_2(r)\underline{\sigma}_1 \cdot \underline{\sigma}_2 + V_s(r) S_{12}$$

This gives the most general interaction between two protons. (due to Wigner) The following assumptions have been made.

1) The interaction potential is independent of orientation in space. This means only scalar products can be admitted.

2) The interaction potential must be invariant to an inversion. (change in the sign of all the

coordinates). There are no other forces in nature which depend on inversion. In our case inversion esp to $r \rightarrow -r$. This means terms like $\underline{\sigma}_i \cdot \underline{r}$ which change sign must be absent. $\underline{\sigma}$ is an angular momentum which is invarient to inversion.

The only non trivial operators which one can construct out of σ_1, σ_2 which are consistant with our assumptions are those incorporated in our expression for the generalized potential. S_{12} is so constructed that it averages to zero over all directions

The Schrodinger equation with the generalized potential inserted is:

$$\frac{\hbar^2}{2m} \nabla^2 \Psi + (E - V(r, \underline{\sigma}_1, \underline{\sigma}_2)) \Psi = 0$$

V is invariant to rotation in space. This means the state of

the system can be described in terms of the total ang. mom. J or its components M.

$$\underline{M} = \underline{L} + \underline{S}$$

The orbital angular mom L or the spin S taken individually are not good quantum numbers, since a rotation of one without a corresponding rotation of the other changes the sign of V. $\underline{L}$ and $\underline{S}$ do not commute with V.

By accident it turns out that for particles of spin $1/2$, S^2 commutes with V [H] and hence is a suitable quantum number.

$$\underline{S}^2 = (\underline{\sigma}_1 + \underline{\sigma}_2)^2 = 6 + 2\,\underline{\sigma}_1 \cdot \underline{\sigma}_2$$

This commutes with everything except possibly S_{12}. It can be shown that for two particles of spin $\frac{1}{2}$

$$\underline{\sigma}_1 \cdot \underline{\sigma}_2 \, S_{12} - S_{12} \, \underline{\sigma}_1 \cdot \underline{\sigma}_2 = 0$$

We have $\underline{\sigma}_1 \cdot \underline{\sigma}_2 = 1$ for $S=1$

$\underline{\sigma}_1 \cdot \underline{\sigma}_2 = -3$ " $S=0$

also

S_{12} operating on a wave function of spin zero $= 0$ [Check on the function $\psi(r)[\alpha(1)\beta(2) - \alpha(2)\beta(1)]$]

Since S_{12} is not zero only for the single state $S=1$ for which $\underline{\sigma}_1 \cdot \underline{\sigma}_2 = 1$, S_{12} obviously commutes with $\underline{\sigma}_1 \cdot \underline{\sigma}_2$. If the spins of the two particles were 1, the possible spin states would be

$S = 0, 1, 2$.

In this case, we have two spin states for which $S_{12} \neq 0$ and hence S_{12} no longer commutes with $\underline{\sigma}_1 \cdot \underline{\sigma}_2$.

5/12/47

I We have seen that the Hamiltonian must be invariant to inversion, $r \to -r$. This property is called parity and

may be used to characterize the wave function.

We have:

$$H(r)\,\psi(r) = E\,\psi(r)$$

$$H(-r)\,\psi(-r) = E\,\psi(-r)$$

$H(r) = H(-r)$ $\therefore$ $H(r)\,\psi(-r) = E\,\psi(-r)$

$\psi(-r)$ satisfies the same differential equation as $\psi(r)$. If there is no degeneracy, this implies

$$\psi(-r) = k\,\psi(r)$$

$$\psi(r) = k^2\,\psi(r)$$

$$k^2 = 1$$

2b $k = +1$ ψ has even parity

$k = -1$ ψ has odd parity

In the case of Hydrogen, we have the same E for different values of l, hence the wave function need not have parity.

We can use the parity of the wave function to show that

an {molecule / atom} in a definite non degenerate state has no dipole moment.

$$\underline{D} = \int \psi^*(r)\, \underline{r}\, \psi(r)\, d\tau$$

$$= \int \psi(-r)\, -\underline{r}\, \psi(-r)\, d\tau = \int \psi^*(r)\, -\underline{r}\, \psi(r)\, d\tau$$

$$= -\underline{D} \quad \text{Therefore } \underline{D} = 0.$$

It is true that a NaCl molecule in a fixed position has a dipole moment, however, the rotation of the molecule in a given quantum state causes this moment to disappear on the average. Hydrogen is an exception.

Parity - Orbital Momentum

$$\psi = \phi(r)\, P_L^M(\theta)\, e^{iM\phi}$$

Inversion $r \to -r$ esp to

$$\theta \to \pi - \theta$$
$$\phi \to \pi + \phi$$

We obtain:

$$\psi(-r) = (-1)^L \psi(r)$$

For L even - even parity
For L odd - odd parity
The parity is a good quantum number.

For a given value of J

$S = 0$	$L = J$	
$S = 1$	$L = J$	or $L = J \pm 1$
$S = 2$	$L = J, J \pm 2$	or $L = J \pm 1$
...	...	...

Let us consider a few special cases.

	$S=0$	$S=1$	
$J = 0$	^1S	$^3P_0$	
$J = 1$	^1P	$^3P_1$	$^3S_1 + {}^3D_1$
$J = 2$	^1D	$^3D_2$	$^3P_2 + {}^3F_2$

Ref.

Schwinger & Rarita PR 59 436 { Special potential
Hepner & Peierls PRS 181 43 { General

In order to compute the coupling between the S and D states for the deuteron, Schwinger & Rarita assume a potential of the form

$$V = -J(r)\{1 - \tfrac{1}{2}g + \tfrac{1}{2}g\,\underline{\sigma}_1 \cdot \underline{\sigma}_2 + \delta S_{12}\}$$

g and δ are constants to be determined empirically. V is so constructed that in the singlet state the δ term disappears and in the triplet state the g term disappears. Our interest is in the triplet state. (The deuteron has no stable singlet state)

The wave function is:

$$\psi = \frac{u(r)}{r} + \frac{w(r)}{r} P_2(\theta\phi)$$

where

1) $$\frac{d^2u}{dr^2} + \frac{M}{\hbar^2}(E + J(r))u = -\sqrt{8}\,\delta \frac{M}{\hbar^2} J(r) w$$

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2) $$\frac{d^2w}{dr^2} + \frac{M}{\hbar^2}(E + J(1-2\delta))w - \frac{6}{r^2}w = -\sqrt{8}\,\delta \frac{M}{\hbar^2} J(r) u$$

Equation 1) shows coupling of the S state to the D state thru δ. Equation 2) shows a modified potential for the D state also coupling to the S state. The coupling is the same in both cases.

Assume $J(r)$ a square well

$$J(r) = V_0 \quad r < a$$
$$= 0 \quad r > a$$

For each value of δ the equations can be integrated numerically to give u, w and the energy E. There are four arbitrary constants of integration which may be used to satisfy the 4 boundary conditions $u = w = 0$ at $r = 0$ and ∞. The available constants in the potential are:

$$V_0, a, \delta.$$

The pertinent experimental information available is the energy and the

quadrapole moment. This will enable us to fix two of the constants in the potential.

We arbitrarily assume

$$a = 2.8 \times 10^{-13} \text{ cm.}$$

For a given V_0 we can determine γ from the binding energy. The quadrapole moment is then determined from the relation

$$Q = \int (3z^2 - r^2) \psi^2 d\tau$$

where the spin is in the Z direction The spherically symmetric term will give no contribution to the integral.

$$Q = \int 2r^2 P_2(\theta) [\mu w P_2 + w^2 P_2^2] 4\pi dr$$

V_0	γ	Q
20	0	0
13.8	.775	2.73×10^{-13} cm^2

5/16/47

I Results of Schwinger & Rarita's calculation of the tensor force for the deuteron.

Interaction assumed

$$-J(r)\left(1 - \tfrac{1}{2}g + \tfrac{1}{2}g\,\sigma_1 \cdot \sigma_2 + \gamma S_{12}\right)$$

where

$$J(r) = V_0 \quad r < a$$
$$= 0 \quad r > a$$

Assume $a = 2.8 \times 10^{-13}$ cm

V_0 ~ central force γV_0 ~ tensor force.

V_0 / 2.18 Mev	γV_0 / 2.18	Q millibarns
9.78	0	0
6.40	4.96	2.73
3.57	7.14	3.34
0	9.42	3.71
-4.	12.0	4.05

The observed value of Q is 2.73 millibarns. This is approximately 1/2 the maximum possible value

for any mixture of S and D states. (6 millibarns)

The problem can also be treated as a perturbation of spherically symmetric S and D states by a tensor potential.

For the observed value of Q the fraction of D state computed from the formula given in the last lecture is: 3.9%. The magnetic moment gives 4%. Bethe feels the agreement is fortuitous for two reasons.

First: The result is sensitive to the shape of the potential well. A potential of the form

$$J(r) = A e^{-\mu r}/r \quad ; \mu = \frac{mc}{\hbar}$$

gives 6% D state.

Second: A relativity correction must be applied, since a particle in a deep potential hole has a large K.E.

A calculation of the relativity correction has been carried out by several people. The most recent papers are those of Sachs and Schwinger.
1. Sachs finds the result depends on the assumption made about relativity. He gives two values for the correction $+2\%$ and -2%
2. Schwinger finds the result is independent of the assumptions made and gives a value -1%.

Let us now investigate the ^1S state.

The interaction is

$$J(r)(1-2g)$$

From n-p scattering we have

$$11.9 = V_0(1-2g)$$

From the ^3S state

$$14 = V_0$$

Assuming the central force in

the singlet and triplet state is the same

$$g = .07 .$$

It turns out that by a proper choice of the range, a, of the nuclear forces, we can make $g = 0$. This is satisfying esthetically but there is no other reason for it. The value of "a" which does this is:

$$a = 2.7 \times 10^{-13} \text{ cm}$$

We have derived a fairly strong tensor force. We must now investigate to find out if this force is reconcilable with the body of knowledge we have about the deuteron.

1. n - p scattering
   Ref Hepner & Peierls PRS 181 43

It is found that for energies

below 5 Mev. the tensor force does not effect the n-p scattering x-sections appreciably. The table below gives a typical comparison of values:

E	σ C + Tensor	σ Central
0	4.21	4.31
2.82	2.53	2.56

Angular distribution

$$1 + .0017 \cos^2\theta$$

2. Radiative Capture

The radiative capture x-section is proportional to

$$\int \psi_{gr}\, \psi_{1s}\, d\tau$$

The only contribution comes from the S part of the ground state wave function. The value of the integral will be changed by the normalization of S part of the wave function.

	σ C+Tensor	σ Central
.025 ev	.302	.312

3. Photo Electric disintegration.
This will again be changed by the normalization of the wave function.

	σ C+T	σ C.
.44 Mev	1.199	1.231

Angular distribution
$\sin^2\theta + .0007$

4. Photo Mag. disintegration
.44 Mev .328
Ang. dis $\sin^2\theta + .182$

Saturation of Nuclear Forces.

We have seen that the binding energy of nuclii is approx. proportional to A. This implies saturation of the nuclear forces.

Let us investigate first the case of ordinary forces.

i.e. $V = V(r, \underline{\sigma}_1 \cdot \underline{\sigma}_2, S_{12})$

We can get an upper limit to the lowest energy state by application of the variational method

$$W = \frac{\int \psi^* H \psi \, d\tau}{\int \psi^* \psi \, d\tau}$$

Assume a sphere of radius R for the nucleus. Each particle in the nucleus must be represented by a spherical wave which vanishes at the surface of the sphere. We can assume for the wave function a product of the wave functions for the individual particles.

$$\psi_{\text{nucleus}} = \prod \psi_{\text{nucleons}}$$

For central forces we obtain $W = W(R)$. If we minimize W as a function of R we find that for $A > 50$ W becomes very large. For $A = 238$, $W = -238$ Bev. This is more than enough energy to wipe out a Plutonium nucleus. This is nonsense and we see that we cannot have ordinary forces.

Ref: Volkoff PR 62, 126.

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I Saturation of Nuclear Forces (cont.)

Ordinary forces do not lead to saturation.

$$W = \frac{\int \psi^* H \psi}{\int |\psi|^2} = T + V$$

If we assume a plane wave to represent each particle in

the nucleus. Then the kinetic energy of the particles will be equal that of a Fermi gas of the same no. of particles, A.

$$T = \dots A\left(\frac{A^{1/3}}{R}\right)^2 \sim \frac{A^{5/3}}{R^2}$$

where R = nuclear radius

The potential energy is proportional to the number of pairs of interacting particles times a function of R

$$V \sim A^2 f(R)$$

Taking the ratio

$$V/T = A^{1/3} f(R) R^2$$

The factor $f(R) R^2$ has the form $V_0 a^2$. If we assume $f(R)$ behaves like V_0, then $f(R) R^2$ is an insensitive function of R. For some value of A, $V/T \gg 1$. For larger A, the BE will

increase as A^2 and this is contradictory to experiment. Ref. R.M.P. Part A.

Our problem now will be to construct a potential which leads to saturation of forces. The average potential between two particles must not be attractive or the B.E. of the nucleus will vary as A^2. Let us try

$$V = \underline{\sigma}_1 \cdot \underline{\sigma}_2 \, J(r)$$

If $J(r) < 0$, V will be attractive for parallel spins. Since it is possible to have the spins of all particles parallel this leads to a potential between particles which is attractive on the average and hence we do not get saturation.

If $J(r) > 0$, V attractive in the singlet state (spins antiparallel) and repulsive in the triplet state (spins parallel). Since it is impossible to each spin antiparallel to all the rest, we have the possibility of a potential which is not attractive on the average. Wigner has shown that this potential does lead to saturation.

This potential contradicts the results of our experiments on the deuteron. The deuteron potential is attractive in both the singlet and triplet states. We have in fact attraction in each of the following cases

$L = 0$	$N - P$	$\uparrow\uparrow$
	$N - P$	$\uparrow\downarrow$
	$P - P$	$\uparrow\downarrow$

This contradictory evidence:

a. Saturation

b. Attractive forces for $L = 0$ leads one to suspect that the value of L may be associated with the phenomena of saturation.

Let us examine the heavier nuclei. $H^3 + He^3$, He^4 are all bound. (He^4 is very strongly bound) He^5 and Li^5 are not bound, hence a repulsive component must come into play when a fifth nucleon is added.* He^4 is completely symmetrical in the space coordinates. He^5 and Li^5 have antisymmetrical space functions. If we follow this

* Note: This 5th nucleon must be added in an $n = 2$ state. The node in the wave function means an increased K.E. It is extremely doubtful, however, whether this increased K.E. could account for the unstability of He^5 & Li^5

hint we may say

a. Repulsion means

Antisymmetry → Odd parity → Odd l

b. Attraction

Symmetry → Even parity → even l

Exchange Forces

We assume that an anti symmetrical spatial wave function (for interchange of two particles) means repulsion.

1. Ordinary, Wigner Forces

$$\frac{\hbar^2}{2m}\sum_j \nabla_j^2 \psi + E\psi(\underline{r}_1 M_1, \underline{r}_2 M_2)$$

$$= V\psi(\underline{r}_1 M_1, \underline{r}_2 M_2)$$

2. Exchange, Majorand Forces

$$= V\psi(\underline{r}_2 M_1, \underline{r}_1 M_2)$$

In case 2, a symmetrical spatial wave function means we have

ordinary forces with an effective potential V, attraction. The antisymmetrical spatial wave functions means an effective potential $-V$, repulsion.

We shall now show that if we assume the above type of force a given proton can attract strongly one other proton and two neutrons. This follows from the fact that these are just the particles which can have the same spatial wave functions as the original proton and hence be symmetric to interchange of the particles. This structure of 2 protons and 2 neutrons is the alpha particle. We might expect the BE/particle for heavier nuclei to be about the same as that for the alpha particle. This is

approx. so.

H	B.E	8×10^{-3} mass units
α		10^{-3}
Heavy average		0

Two other types of nuclear forces have been postulated.

3. Spin Exchange, Bartlett

$$= V \Psi(\underline{r}_1, M_2, \underline{r}_2 M_1)$$

4 Heisenberg

$$= V \Psi(\underline{r}_2 M_2, \underline{r}_1, M_1)$$

Both the Bartlett and Heisenberg forces contradict exp. on the deuteron. This does not mean that they may not be mixed in with the other forces, however.

The Bartlett force has an

effective potential

$+V \qquad S = 1$

$-V \qquad S = 0$

We can express the Bartlett force as follows

$$V = V(r)\,\tfrac{1}{2}(1 + \underline{\sigma}_1 \cdot \underline{\sigma}_2)$$

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We define an operator

$\frac{1}{2}\tau_z = +\frac{1}{2}$ for a proton

$= -\frac{1}{2}$ for a neutron

The charge of a nucleon is given by $\frac{1}{2} + \frac{1}{2}\tau_z$. We can describe a system of nucleons in terms of the coordinates $\underline{r}$ u & τ_z

We define also a charge wave function γ for a proton

δ for a neutron

The wave function for a system of 2 protons in our new notation will be:

$$\gamma(1)\,\gamma(2)\,\underbrace{\psi(\underline{r}_1, u_1, \underline{r}_2, u_2)}_{\text{antisymm.}}$$

Deuteron wave function. In our new notation we regard the proton and neutron as two states of the nucleon. Since the total wave function must be antisymm. we have the following possibilities

ψ (spin, space)	χ (charge)
antisymm.	symm.
symm	antisymm

$$\chi_{sym} = \frac{1}{\sqrt{2}}\{\alpha(1)\beta(2) + \beta(1)\alpha(2)\}$$
$$\chi_{anti} = \frac{1}{\sqrt{2}}\{\alpha(1)\beta(2) - \beta(1)\alpha(2)\}$$

By analogy with the spin, we define an <u>isotopic spin</u> which is $T = 1$ for χ_{sym} and $T = 0$ for χ_{anti}

For a system of A nucleons we have:

$$T_z = \frac{1}{2}\sum_1^A T_{z_i}$$

$$Z = T_z + \frac{1}{2}A$$

Exchange Operators

1) Position P
2) Spin Q
3) Charge R

$$PQR\Psi = -\Psi$$
$$P^2 = Q^2 = R^2 = 1$$

P Q R commute

$$P = -QR$$

By analogy:

Spin $Q = \frac{1}{2}(1 + \underline{\sigma}_1 \cdot \underline{\sigma}_2)$

Charge $R = \frac{1}{2}(1 + \underline{\tau}_1 \cdot \underline{\tau}_2)$

$\underline{\tau}_1 \cdot \underline{\tau}_2 = 1$ $\quad T = 1$
" $= -3$ $\quad T = 0$

$\tau_z \gamma = \gamma$ $\quad \tau_x \gamma = \delta$ $\quad \tau_y \gamma = i\delta$
$\tau_z \delta = -\delta$ $\quad \tau_x \delta = \gamma$ $\quad \tau_y \delta = -i\gamma$

$$(\tau_x + i\tau_y)\delta = 2\gamma$$
$$(\tau_x + i\tau_y)\gamma = 0$$

The Majorana force can be expressed in our new notation as follows

$$P = -\frac{1}{4}(1 + \underline{\sigma}_1 \cdot \underline{\sigma}_2)(1 + \underline{\tau}_1 \cdot \underline{\tau}_2)$$

We can express a general force in terms of the operators $\underline{\sigma}_1 \cdot \underline{\sigma}_2$ and $\underline{\tau}_1 \cdot \underline{\tau}_2$. The tensor exchange will be given by

$$(1 + \underline{\tau}_1 \cdot \underline{\tau}_2) S_{12}$$

This follows from the fact that $\underline{\sigma}_1 \cdot \underline{\sigma}_2$ on S_{12} reproduces S_{12}.

[Wigner 1937-38 Theory of medium light nuclei $A = 3-50$]

Let us now investigate the following symmetric potential

$$(\underline{\sigma}_1 \cdot \underline{\sigma}_2)(\underline{\tau}_1 \cdot \underline{\tau}_2) V(r)$$

For ~equal numbers of protons and neutrons both these factors

average to zero and we do not get a contradiction to saturation.

Assume $V > 0$

Deuteron:

	spin $\sigma_1 \cdot \sigma_2$	space L	T	$\tau_1 \cdot \tau_2$
^3S	sym $+1$	sym 0	0	-3
^1S	anti -3	sym 0	1	$+1$
He^2				
^1S	anti -3	sym 0	1	$+1$

In all of the above cases we have the same central force (assuming the sym potential). That this was the case was suspected by Schwinger & Rarita. It was this fact which led Condon & Cassen to invent the isotopic spin.

$He^2 H^2$

^3P	sym $+1$	anti 1	1	$+1$
H^2				
^1P	anti -3	anti 1	0	-3

In the ^3P state the potential is repulsive +1. A Majorand potential gives +3.

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We have postulated three types of forces

I Sym. $\underline{\tau}_1 \cdot \underline{\tau}_2 (\underline{\sigma}_1 \cdot \underline{\sigma}_2 + \gamma S_{12})$

II Majorand $(1 + \underline{\tau}_1 \cdot \underline{\tau}_2)(1 + \underline{\sigma}_1 \cdot \underline{\sigma}_2 + \gamma S_{12})$

III Ordinary $(1 + \gamma S_{12})$

These forces are interpreted in terms of meson exchange as follows:

III Neutral mesons

$$N \rightarrow N + u^\circ \qquad P + u^\circ \rightarrow P$$

This gives rise to forces only between pairs of protons and pairs neutrons which violates our assumption that the forces between nucleons are independent of charge.

II Charged mesons

$N \rightarrow P + \mu^- \quad ; \quad P + \mu^- \rightarrow N$

This gives a force only between Neutron and proton which again contradicts our assumption about the symmetry of nuclear force.

I Symmetric Theory

This theory includes both neutral and charged mesons and gives the symmetric forces we want.

Rarita & Schwinger made the following comparison of these forces.

For a square well with a $= 2.8 \times 10^{-13}$ cm., the effective potentials in the S state are:

$^1S \sim 11.9$ Mev

$^3S \sim 20$ Mev

The potentials in the p state

are related to those in the
s state as follows

^1P $\quad -3V_s$

^3P $\quad -\frac{1}{3}V_s$

The neutral theory, III gives the following potentials for the ^3p state

$^3P_0$ $\quad +29$ Mev

$^3P_1$ $\quad -35$ Mev

$^3P_2$ $\quad -9.6$ Mev

(The subscript gives the value of J.) The fact that states of higher L may be mixed with these does not effect the potential because the particles are far apart.

The table below gives the phase shift and the differential x-sections for the three theories:

	η_0	η_1	η_2	$\sigma(\theta)$
I Sym	.074	-.054	-.017	$\lambda^2(.004 + .005\cos^2\theta)$
II Charged	.513	.115	.046	.103 -.002 "
III Neutral	.102	.995	.073	.487 +.687 "

III Gives a large $^3P_1$ scattering. The tensor force introduces an isotropic term.

II Isotropic scattering.

I Small P scattering.

If we added the S scattering to the P scattering, we obtain:

	σ_{total}	$\sigma(\theta)$
I	.621	$1 - .080\cos\theta + .077\cos^2\theta$
II	.666	$1 + .126$ " $+ .042$ "
III	.983	$1 + .932$ " $+ .457$ "

III Gives a strong forward maximum due to interference of $^3P_1$ and ^1S scattering.

Observations

The total x section is the thing most easily observed, but this turns out to be quite sensitive to the value of the range of nuclear forces assumed.

Amaldi has measured the angular distribution of neutrons scattered by protons in paraffin. His results are in agreement with the neutral theory. Nobody believes the neutral theory however, and new results seem to indicate the scattering may be isotropic as it is in the symmetric theory. (Champion & Powell)

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Proton - Proton Scattering

References

Wilson PR 71 339
71 384

Scattering of protons and neutrons at very high energies - 100 Mev

We can apply the Born approx. to this problem.

$$d\sigma \sim \left| \int V(r) e^{i\vec{q}\cdot\vec{r}} d\tau \right|^2$$

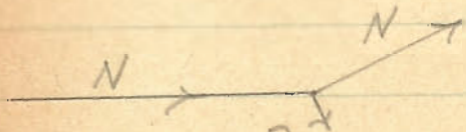
where $\vec{q} = \frac{\vec{p}_0 - \vec{p}}{\hbar}$

For $p/\hbar a \gg 1$, the scattering is large only for small angles. This condition is fullfilled in our case: $\hbar/p = \lambda\!\!\!^{-} = .6 \times 10^{-13}$, $a \sim 2.8 \times 10^{-13}$.

$$q \sim 2k_0 \sin\theta/2 \approx \frac{p\theta}{\hbar}$$

$$qa = a\theta/\lambda\!\!\!^{-}$$

The scattering will be large for $qa \approx 1$, $\theta \approx \lambda\!\!\!^{-}/a$. This is the classical scattering angle. For 100 Mev protons $\theta \sim .2$ radian $= 10° - 15°$



The energy of the recoil proton will be

$$E_p = \frac{(\vec{p}_0 - \vec{p})^2}{2M} = \frac{(\hbar q)^2}{2M} \approx \frac{\hbar^2}{2Ma^2}$$

This energy of the recoil proton is closely related to the depth of the potential well

$$V_0 a^2 \approx \frac{\pi^2}{4} \hbar^2 / M$$

$$V_0 \approx \frac{\hbar^2}{2Ma^2} \doteq E_p$$

If we have exchange forces, the scattering is the same as for ordinary forces except that the proton and neutron exchange places, the proton gets almost all the energy and the neutron recoils with $E \approx V_0$. Experiments have been performed at Cal with 100 Mev neutrons scattered by paraffin. They show a large number of 100 Mev protons in the forward direction. This indicates a large percentage of exchange forces, but does

not distinguish between
Symmetrical and Majorand
forces.

Symmetrical interaction

$$(\underline{\tau}_1 \cdot \underline{\tau}_2)(\underline{\sigma}_1 \cdot \underline{\sigma}_2) V$$

$S = 1$	V	even ℓ
	$-\frac{1}{3} V$	odd ℓ

Ordinary forces $\sim f(\theta)$
Exchange forces $\sim f(\pi - \theta)$

even ℓ	$\frac{1}{2}(f(\theta) + f(\pi-\theta))$
odd ℓ	$\frac{1}{2}(f(\theta) - f(\pi-\theta))$

To get ordinary amplitude add.
" " exchange " subtract.

even ℓ $-\frac{1}{3}$ odd ℓ

$S = 1$

Scattered amp $= \frac{1}{3} f(\theta) + \frac{2}{3} f(\pi-\theta)$

$S=0$ odd L $-3V$

even L V

Scattered amp $= f(\theta) + 2f(\pi-\theta)$

$$\sigma \sim \frac{3}{4}[S=1]^2 + \frac{1}{4}[S=0]^2$$

$$= \frac{1}{3}f^2(\theta) + \frac{4}{3}f^2(\pi-\theta) - \frac{2}{3}f(\theta)f(\pi+\theta)$$

$f(\theta)$ large only for $\theta \leq .2$
The interference is unimportant.
In the lab. system there will be maxima at 0 & $90°$
Many high energy protons in the forward direction, $\frac{4}{3}f^2(\pi-\theta)$
Fewer low energy protons at $90°$ $\frac{1}{3}f^2(\theta)$. The $90°$ protons are practically impossible to detect.

For proton-proton scattering the ordinary and exchange (Majorana) forces give the same

results.

$$[\, f^2(\theta) + f^2(\pi-\theta) - f(\theta) f(\pi-\theta) \,]$$

The symmetrical interaction gives

$S = 0$ $\quad [\, f^2(\theta) + f^2(\pi-\theta) - f(\theta) f(\pi-\theta) \,]$

$S = 1$ $\quad -\frac{1}{3} V$

$$\frac{1}{3} [\, f^2(\theta) + f^2(\pi-\theta) - f(\theta) f(\pi-\theta) \,]$$

This modification of the triplet scattering by the factor $\frac{1}{3}$ should make it possible to distinguish between Majorana and symmetric forces from proton-proton scattering.