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Advanced Q.M.

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NATIONAL

Made in U. S. A.

THIS BOOK CONTAINS EYE-EASE PAPER

"Easy on the Eyes"

Advanced Q M

R. P. Feynman Fall 1948

Dirac Eq:

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = H\psi$$

where: $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$

$$H = c\alpha\cdot\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right) + e\phi + \beta mc^2$$

$$\alpha = \begin{pmatrix}0 & \sigma\\ \sigma & 0\end{pmatrix} \quad \beta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

We may also write

$$-\frac{\hbar}{i}\frac{\partial \psi_a}{\partial t} = c.\sigma\cdot\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\psi_b + e\phi\psi_a + mc^2\psi_a$$

$$-\frac{\hbar}{i}\frac{\partial \psi_b}{\partial t} = c\sigma\cdot\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\psi_a + e\phi\psi_b - mc^2\psi_b$$

where

$$\psi_a = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \psi_b = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$$

Commutation relations:

$$\alpha_x \alpha_y + \alpha_y \alpha_x = 0$$

$$\alpha_x \beta + \beta \alpha_x = 0$$

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$$

Problem: Show that if Ψ is replaced by $\Psi' = S\Psi$ ($\det S \neq 0$) then Ψ' satisfies a D.E. with

$$\alpha' = S\alpha S^{-1}, \quad \beta = S\beta S^{-1}$$

Proof:

$$-\frac{\hbar}{i}\frac{\partial(S^{-1}\Psi')}{\partial t} = H S^{-1}\Psi'$$

$$-\frac{\hbar}{i}\frac{\partial\Psi'}{\partial t} = (SHS^{-1})\Psi'$$

Relation of α's to spin matrix

$$\alpha_x \alpha_y = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{pmatrix} = \begin{pmatrix} i & & & \\ & -i & & \\ & & i & \\ & & & -i \end{pmatrix} = i\sigma_z$$

$$\alpha_y \alpha_z = i\sigma_x$$

$$\alpha_z \alpha_x = i\sigma_y$$

where: $\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$

The α's & β are <u>Hermitian</u>

A matrix A is Hermitian if

$$((A))_{ij}^{*} = ((A))_{ji}$$

$$A^{*} = \tilde{A}$$

If A Hermitian

$$[A\phi]^{*} = \phi^{*}A^{H} = \phi^{*}A$$

If A, B Hermitian

$$[(AB + BA)\phi]^{*} = (B\phi)^{*}A + (A\phi)^{*}B$$

$$= \phi^{*}(BA + AB)$$

$$[i(AB - BA)\phi]^{*} = (iB\phi)^{*}A - (iA\phi)^{*}B$$

$$= -i\phi^{*}(BA - AB)$$

hence $AB + BA$ & $i(AB - BA)$ are Hermitian but in general AB is not Hermitian.

If AB commute ie $AB = BA$ then AB Hermitian

If AB anticommute ie $AB = -BA$ then iAB Hermitian

In order to show relativistic invariance of the Dirac Eq. the following matrices are nice:

$$\gamma_x = \beta \alpha_x$$
$$\gamma_y = \beta \alpha_y$$
$$\gamma_z = \beta \alpha_z$$
$$\gamma_t = \beta$$

The γ's form a 4 vector. They are <u>not</u> Hermitian.

Show: $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}$

where: $g_{\nu\mu} = 0 \quad \mu \neq \nu$ i.e. γ's anticommute

$$g_{11} = g_{22} = g_{33} = -g_{44} = -1$$

Proof: a) $\beta \alpha_x \beta \alpha_y + \beta \alpha_y \beta \alpha_x$

$= -\alpha_x \alpha_y - \alpha_y \alpha_x = 0$

b) $\beta \beta \alpha_x + \beta \alpha_x \beta$

$= \alpha_x - \alpha_x = 0$

c) $\beta \alpha_x \beta \alpha_x + \beta \alpha_x \beta \alpha_x$

$= -\alpha_x^2 - \alpha_x^2 = -2$

d) $\beta \beta + \beta \beta = 2$

$$\gamma_5 = \gamma_x \gamma_y \gamma_z \gamma_t = \alpha_x \alpha_y \alpha_z$$
$$\gamma_\mu \gamma_5 + \gamma_5 \gamma_\mu = 0$$

Continuity Equation

$$\overleftarrow{\psi}\,// + \frac{\hbar}{i}\frac{\partial \psi^*}{\partial t} = [(c\alpha\cdot(p-\tfrac{e}{c}A) + e\phi + \beta mc^2)\psi]^*$$
$$-\overrightarrow{\psi^*}// - \frac{\hbar}{i}\frac{\partial \psi}{\partial t} = [(c\alpha\cdot(p-\tfrac{e}{c}A) + e\phi + \beta mc^2)\psi]$$

add

$$\frac{\hbar}{i}\frac{\partial}{\partial t}(\psi^*\psi) = -c\alpha\cdot\frac{\hbar}{i}(\nabla\psi^*)\psi - c\alpha\cdot\psi^*\frac{\hbar}{i}\nabla\psi$$

$$\frac{\partial}{\partial t}(\psi^*\psi) + c\,\nabla\cdot(\psi^*\alpha\psi)$$

Time Derivative

$$\langle\dot{A}\rangle = \frac{d}{dt}\int\psi^* A\psi\,d\tau$$
$$= \int\left[\frac{\partial\psi^*}{\partial t}A\psi + \psi^* A\frac{\partial\psi}{\partial t} + \psi^*\frac{\partial A}{\partial t}\psi\right]d\tau$$
$$= \frac{i}{\hbar}\int[(H\psi)^* A\psi - \psi^* AH\psi + \psi^*\frac{\partial A}{\partial t}\psi]\,d\tau$$

$$\langle\dot{A}\rangle = \frac{i}{\hbar}\left\langle(HA - AH) + \frac{\partial A}{\partial t}\right\rangle$$

$$\left(H = c\alpha\cdot(\tfrac{\hbar}{i}\nabla - \tfrac{e}{c}A) + e\phi + \beta mc^2\right)$$

$$\dot{x} = \frac{i}{\hbar}(Hx - xH) = c\alpha_x$$

$$v = c\alpha$$

Force Equation

$$\dot{p}_x = \frac{i}{\hbar}\left(H\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}H\right) = e\frac{\partial}{\partial x}(\alpha\cdot A) - e\frac{\partial\phi}{\partial x}$$

$$\dot{p} = e\,\nabla(\alpha\cdot A) - e\nabla\phi$$

$$\frac{d}{dt}\frac{e}{c}A = \frac{e}{c}\left(\frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z} + \frac{\partial A}{\partial t}\right)$$
$$= \frac{e}{c}\left[(c\alpha\cdot\nabla)A + \frac{\partial A}{\partial t}\right]$$

$$\frac{d}{dt}\left(p - \frac{e}{c}A\right) = e\left[\alpha\times(\nabla\times A) - \left(\nabla\phi + \frac{1}{c}\frac{\partial A}{\partial t}\right)\right]$$

$$= e\left[\frac{v}{c}\times B + E\right] = ef$$

Torque Equation

$$\frac{d}{dt}L = \frac{d}{dt}\left(R\times\left(p - \frac{e}{c}A\right)\right) = \dot{R}\times\left(p - \frac{e}{c}A\right)$$
$$+ R\times\frac{d}{dt}\left(p - \frac{e}{c}A\right)$$
$$= c\alpha\times\left(p - \frac{e}{c}A\right) + R\times ef$$

$$\frac{d}{dt}\sigma_z = \frac{d}{dt}\frac{1}{i}\alpha_x\alpha_y = \frac{i}{\hbar}\left(H\alpha_x\alpha_y - \alpha_x\alpha_y H\right)$$
$$= -\frac{2}{\hbar}\left[\alpha_x\left(p_y - \frac{e}{c}A_y\right) - \alpha_y\left(p_x - \frac{e}{c}A_x\right)\right]$$
$$= -\frac{2}{\hbar}\left[\alpha\times\left(p - \frac{e}{c}A\right)\right]_z$$

$$\frac{d}{dt}\left(L + \frac{\hbar}{2}\sigma\right) = e\,R\times f$$

Let:

$$\pi = (p - \tfrac{e}{c}A)$$
$$W = H - e\phi$$
$$E = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

Then

$$i\hbar\dot{\alpha} = -2\pi + 2W\alpha = +2\pi - 2\alpha W$$

$$\dot{W} = \alpha \cdot E$$

$$\frac{d}{dt}\left(x - \frac{i\hbar}{2mc}\beta\alpha\right) = \frac{\beta\pi}{mc}$$

$$\frac{d}{dt}\left(t - \frac{i\hbar}{2mc}\beta\right) = \frac{\beta W}{mc}$$

$$\frac{d}{dt}\left(i\alpha_x\alpha_y\alpha_z\right) = \frac{2mc}{\hbar}(\beta\alpha_x\alpha_y\alpha_z)$$

$$\frac{d}{dt}(\beta\sigma) = 2(\beta\alpha_x\alpha_y\alpha_z)\pi$$

Relativistic Invariance

$$[c\alpha\cdot(p-\tfrac{e}{c}A)+(\tfrac{h}{i}\tfrac{\partial}{\partial t}+e\phi)]\Psi=-\beta mc^2\Psi$$

$$[-c\beta\alpha\cdot(p-\tfrac{e}{c}A)-\beta(\tfrac{h}{i}\tfrac{\partial}{\partial t}+e\phi)]\Psi = mc^2\Psi$$

$$\begin{cases}\gamma_{1\,2\,3\,4} = \beta\alpha_x,\ \beta\alpha_y,\ \beta\alpha_z,\ \beta \\ \Pi_{1\,2\,3\,4} = (p-\tfrac{e}{c}A)_{xyz},\ (-\tfrac{h}{i}\tfrac{\partial}{\partial t}-e\phi)\end{cases}$$

We shall use the convention

$$r_1^4\cdot r_2^4 = t_1\cdot t_2 - \underline{r}_1\cdot\underline{r}_2$$

hence we may write

$$(\gamma_\mu\Pi_\mu)\Psi = mc^2\Psi$$

or

$$\tilde{\Psi}(\gamma_\mu\Pi_\mu) = mc^2\tilde{\Psi}$$

where $\tilde{\Psi}\equiv\Psi^*\beta$

$$j_\mu = \Psi^*\alpha_{xyz}\Psi,\quad \Psi^*\Psi$$

$$= \tilde{\Psi}\gamma_\mu\Psi$$

Normalization

$$1 = \int \tilde{\Psi} (N_\mu \gamma_\mu) \Psi \, d\tau$$

where N_μ is a unit vector in the t direction

$$= \int \Psi^* \beta \, \beta \Psi \, d\tau = \int \Psi^* \Psi \, d\tau$$

Anti sym ten of 2nd rank

Angular momentum

$$m_{\mu\nu} = p_\mu x_\nu - x_\mu p_\nu$$

Fields

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

Spin

$$\sigma_{\mu\nu} = i(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)/2$$

3rd rank : 4 components

$$A_\tau = \gamma_\sigma \gamma_\mu \gamma_\nu - \gamma_\mu \gamma_\sigma \gamma_\nu + \gamma_\mu \gamma_\nu \gamma_\sigma$$

Maxwell Eq $\frac{\partial F_{\mu\nu}}{\partial x_\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x_\mu} + \frac{\partial F_{\sigma\mu}}{\partial x_\nu} = 0$

4^{th} rank: Pseudo scalar

$$\gamma_5 = \gamma_x \gamma_y \gamma_z \gamma_t$$

Resolution into large + small components: Nonrelativistic steady state

$$H\Psi = E\Psi = (mc^2 + W)\Psi$$

$$1)\quad c\,\sigma\cdot\pi\,\Psi_b + e\phi\Psi_a + mc^2\Psi_a = (mc^2+W)\Psi_a$$
$$2)\quad c\,\sigma\cdot\pi\,\Psi_a + e\phi\Psi_b - mc^2\Psi_b = (mc^2+W)\Psi_b$$

From 2) $\Psi_b = c\,\sigma\cdot\pi\,\Psi_a / (2mc^2 + W - e\phi)$

Insert in 1)

$$W\Psi_a = (\sigma\cdot\pi)\frac{c^2}{(2mc^2+W-e\phi)}(\sigma\cdot\pi)\Psi_a + e\phi\Psi_a$$

For $W - e\phi \ll 2mc^2$

$$W\Psi_a = \frac{1}{2m}(\pi\cdot\pi)\Psi_a + \frac{e\hbar}{2mc}(\sigma\cdot B)\Psi_a + e\phi\Psi_a$$

This is Pauli Equation.

Also

$$W\psi_a = \frac{c^2}{(\ \ )}(\sigma\cdot\pi)(\sigma\cdot\pi)\psi_a + \frac{c^2}{(\ \ )^2}\frac{\hbar}{i}(\sigma\cdot\nabla V)(\sigma\cdot\pi)\psi_a$$
$$+ V\psi_a$$

$$= \frac{c^2}{(\ \ )}\left[(\pi\cdot\pi) + \frac{e\hbar}{c}(\sigma\cdot B)\right]\psi_a + \ \dots$$

$$= \pi\cdot\frac{c^2}{(\ \ )}\pi\psi_a + \frac{e\hbar c}{(\ \ )}\sigma\cdot B\psi_a$$

$$+ \frac{c^2}{(\ \ )^2}\frac{\hbar}{i}\left[(\sigma\cdot\nabla V)(\sigma\cdot\pi) - \nabla V\cdot\pi\right] + V\psi_a$$

Using $(\sigma\cdot A)(\sigma\cdot B) = A\cdot B + i\sigma\cdot(A\times B)$

$$W\psi_a = \pi\cdot\frac{c^2}{2mc^2+W-V}\pi\psi_a$$

$$+ \frac{e\hbar c}{2mc^2+W-V}\left\{\sigma\cdot B - \frac{c}{e}\frac{(\sigma\times\pi)\cdot\nabla V}{2mc^2+W-V}\right\}\psi_a + V\psi_a$$

Some interesting numbers

$$\frac{e^2}{\hbar c} \doteq \frac{1}{137}$$

$$\frac{\hbar^2}{me} = .528\ \text{Å} \quad \text{Schrodinger Theory O.K.}$$

$$\frac{\hbar}{mc} = \frac{.528}{137} = 3.85 \times 10^{-11} \quad \text{Dirac Theory OK}$$

$$\frac{e^2}{mc^2} = \frac{.528}{137^2} = 3 \times 10^{-13}\ \text{cm} \quad \text{Unknown region}$$

Free Particle Solution of D.E.

Let $c \equiv 1$

$$\psi = u\, e^{-\frac{i}{\hbar}(Et - \underline{p}\cdot\underline{x})}$$

$$-\frac{\hbar}{i}\frac{\partial}{\partial t} u\, e^{-\frac{i}{\hbar}(Et-\underline{p}\cdot\underline{x})} = E u\, e^{-\frac{i}{\hbar}(Et-\underline{p}\cdot\underline{x})}$$

$$= e^{-\frac{i}{\hbar}(Et-\underline{p}\cdot\underline{x})} H u$$

$$Hu = Eu$$

$$(E - \alpha\cdot p - \beta m)u = 0$$

For non-trivial solutions

$$\det(E - \alpha\cdot p - \beta m)$$

$$\det \begin{pmatrix} E-m & -\sigma\cdot p \\ -\sigma\cdot p & E+m \end{pmatrix} = 0$$

$$E^2 - m^2 - (\sigma\cdot p)(\sigma\cdot p)$$
$$= E^2 - m^2 - p^2$$
$$E^2 = m^2 + p^2$$

We may also obtain this result as follows

$$(E + \alpha\cdot p + \beta m)(E - \alpha\cdot p - \beta m)u = 0$$
$$E^2 - (\alpha\cdot p + \beta m)^2 u = 0$$
$$E^2 - p^2 - m^2 = 0$$

There are four liniar ind ortho normal solutions for the u's. We obtain these as follows:

Let $u' = \begin{pmatrix} u_a' \\ u_b' \end{pmatrix}$

1) $(E-m)u_a - (\sigma\cdot p)u_b = 0$

2) $(E+m)u_b - (\sigma\cdot p)u_a = 0$

Consider: $H = \alpha \cdot p + \beta m$

$\frac{\hbar}{i}\langle \dot{\sigma \cdot p}\rangle = [H\sigma\cdot p - \sigma\cdot p H]$

$= (\alpha\cdot p)(\sigma\cdot p) + m\beta\sigma\cdot p - (\sigma\cdot p)(\alpha\cdot p) - (\sigma\cdot p)\beta m$

Lemmas: $(\alpha\cdot p)(\sigma\cdot p) = -i(\alpha_x\alpha_y\alpha_z)(p\cdot p) = (\sigma\cdot p)(\alpha\cdot p)$

$-i\beta\alpha_x\alpha_y = -i\alpha_x\alpha_y\beta$

hence

$[H, \sigma\cdot p] = 0$, and $\sigma\cdot p$ a constant of motion.

$$(\sigma\cdot p)u_a = Nu_a$$
$$(\sigma\cdot p)u_b = Nu_b$$

Inserting in 1 & 2

$$(E-m)u_a = Nu_b$$
$$(E+m)u_b = Nu_a$$

$$(E^2-m^2) = p^2 = N^2$$
$$N = \pm|p| = s\cdot|p|$$

$$(\sigma\cdot\frac{p}{|p|})u = su \quad : \quad s = \pm 1$$

$s = +1$ Let $\underline{v} = \frac{p}{|p|}$

$$\varphi_2(v_x - iv_y) + \varphi_1(v_z - 1) = 0$$
$$\{\ \varphi_1(v_x + iv_y) - \varphi_2(v_z + 1) = 0\ \}$$

$$\phi_2 = + \frac{(V_x + i V_y)}{(V_z + 1)} \phi_1 .$$

Normalization $|\phi_1|^2 + |\phi_2|^2 = 1$

$$|\phi_1|^2 \left\{ 1 + \frac{V_x^2 + V_y^2}{V_z^2 + 2V_z + 1} \right\} = 1$$

$$|\phi_1|^2 = (V_z + 1)^2 / (2 + 2V_z)$$

$$\phi_+ = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{2(1+V_z)}} \begin{pmatrix} 1 + V_z \\ V_x + i V_y \end{pmatrix}$$

$S = -1$

$$\phi_- = \frac{1}{\sqrt{2(1-V_z)}} \begin{pmatrix} 1 - V_z \\ -V_x - iV_y \end{pmatrix}$$

The u_a & u_b must be multiples of the ϕ's

$$u_a = a \phi$$

$$u_b = b \phi .$$

For $E = + E_p = + \sqrt{m^2 + p^2}$

$$S = +1$$

$$(E_p - m) a = p b$$

$$a = \frac{p}{(E_p - m)} b$$

$$u_a^2 + u_b^2 = a^2 + b^2 = 1$$

$$b^2\left\{1+\frac{p^2}{E_p^2-2mE+m^2}\right\} = 1$$

$$b^2 = (E_p-m)^2/(2E_p^2-2mE)$$

$$b = \sqrt{\frac{E_p-m}{2E_p}}$$

$$a = \sqrt{\frac{E_p+m}{2E_p}}$$

For $E = +E_p$

$$s = -1$$

$$b = \sqrt{\frac{E_p-m}{2E_p}}$$

$$a = -\sqrt{\frac{E_p+m}{2E_p}}$$

etc.

		$E=+E_p$	$E=-E_p$
$s=1$ parallel	u_a	$=\sqrt{\frac{E+m}{2E}}\,\phi_+$	$\sqrt{\frac{E-m}{2E}}\,\phi_+$
	u_b	$=\sqrt{\frac{E-m}{2E}}\,\phi_+$	$\sqrt{\frac{E+m}{2E}}\,\phi_+$
$s=-1$ antipar.	u_a	$=-\sqrt{\frac{E+m}{2E}}\,\phi_-$	$-\sqrt{\frac{E-m}{2E}}\,\phi_-$
	u_b	$=\sqrt{\frac{E-m}{2E}}\,\phi_-$	$\sqrt{\frac{E+m}{2E}}\,\phi_-$

The above table contains the complete solution of the D.E. for a free particle

Matrix Elements for a Free Particle

$$\begin{cases} \Psi = u\,e^{-i(Et-\underline{p}\cdot\underline{x})} \\ (\sigma\cdot p)\Psi = s|p|\Psi \end{cases}$$

$$(\Psi^* A \Psi) = u^* e^{+i(\quad)} A u e^{-i(\quad)}$$
$$= u^* A u$$

$$(u^*, u) = 1 \qquad (a)$$

Method of calc. matrix elements:
Consider
$$u^*(H\alpha + \alpha H)u$$
$$= (Hu)^* \alpha u + u^* \alpha H u$$
$$= E u^* \alpha u + E u^* \alpha u = 2E u^* \alpha u$$
but
$$H = \alpha \cdot p + m\beta$$
$$H\alpha + \alpha H = 2p$$

(Note $(\alpha \cdot A)\alpha_x = -\alpha_x(\alpha \cdot A) + 2A_x$)

Since: $$u^* \alpha u = p/E \qquad (b)$$
Similarly:
$$u^* \beta u = m/E \qquad (c)$$

$$u^* \alpha_x \alpha_y \alpha_z \beta u = 0 \qquad (d)$$

$$u^* i \alpha_x \alpha_y \alpha_z u = -\frac{u^*(\sigma \cdot p)u}{E} = -\frac{s|p|}{E} \qquad (e)$$

One may also consider

$$u^*(H\alpha - \alpha H)u \quad = \quad E\, u^*\alpha u \; - E\, u^*\alpha u = 0$$
$$u^*(H\alpha - \alpha H)u = -2\, u^* i\, \sigma \times p u \; + \; 2 u^* \beta \alpha u = 0$$

$$u^* i \beta \alpha u \quad = -u^*(\sigma \times p)u \qquad\qquad (f)$$

$$u^*\{(\sigma\cdot p)\alpha \; - \alpha(\sigma\cdot p)\}u \; = |p|S\, u^*\alpha u \; - |p|S\, u^*\alpha u = 0$$

$$u^*\alpha \times p u \quad = \quad 0 \qquad\qquad (g)$$

or
$$= u^*\alpha u \times p \; = \; \frac{p \times p}{E} \; = \; 0$$

$$u^*\{(\sigma\cdot p)\sigma + \sigma(\sigma\cdot p)\}u \; = \; 2|p|s\; u^*\sigma u$$
$$= \; 2p$$

$$u^*\sigma u \quad = \quad p/s|p| \qquad\qquad (h)$$

$$u^*\beta\sigma u \quad = \quad \frac{m}{E}\, u^*\sigma u \qquad\qquad (i)$$

$$u^*(\sigma \times p)u = -u^* i \beta \alpha u \quad = \quad 0 \qquad\qquad (j)$$

Matrix elements for a free particle in terms of γ's

$$\langle A \rangle \equiv \int \tilde{\Psi} A \Psi d\tau = \int \Psi^* \beta A \Psi d\tau$$

$$\Psi = u e^{-i p_\mu x_\mu}$$

$$(\Sigma_{\mu\nu} p_\sigma)\Psi = (\sigma \cdot p)\Psi = s|p|\Psi$$

$$\langle 1 \rangle = \frac{\mu}{E}$$

$$\langle \gamma_\mu \rangle = p_\mu / E$$

$$\frac{i}{2}\langle \gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu \rangle = + \frac{\mu}{E} (\Sigma_{\mu\nu})_{asa}$$

$$\frac{i}{3}\langle \gamma_\mu \gamma_\nu \gamma_\sigma - \gamma_\mu \gamma_\sigma \gamma_\nu + \gamma_\sigma \gamma_\nu \gamma_\mu \rangle = + \frac{1}{E}(\Sigma_{\mu\nu} p_\sigma)_{asa}$$

where $(\ )_{asa}$ indicates anti sym. average

Matrix elements between states of a free particle.

$$(p_\mu \gamma_\mu)\Psi = m\Psi$$

We consider the case:

$$E, p_z \rightarrow E, -p_z$$

$\xrightarrow{E,\ p_z}$ initial

$\xleftarrow[E\ -p_z]{}$ final

Let

$$E \equiv p, \quad p_z \equiv q$$

$$W_1 \equiv p\gamma_t - q\gamma_z$$

$$W_2 \equiv p\gamma_t + q\gamma_z$$

$$\int \tilde{\Psi}_2 (W_2 - W_1)\Psi_1 d\tau = m\tilde{\Psi}_2\Psi_1 - m\tilde{\Psi}_2\Psi_1 = 0$$

$$W_2 - W_1 = 2q\gamma_z$$

Hence

$$\langle W_2 - W_1 \rangle_{12} = 0$$

$$\langle \gamma_z \rangle_{12} = 0$$

Similarly

$$\left\{ \begin{aligned} \langle W_2 + W_1 \rangle_{12} &= 2m \langle 1 \rangle_{12} \\ \langle \gamma_t \rangle_{12} &= \frac{m}{p} \langle 1 \rangle_{12} \end{aligned} \right.$$

$$\langle W_2 \gamma_x - \gamma_x W_1 \rangle = 0 = 2p \langle \gamma_x \gamma_t \rangle$$

$$\left\{ \begin{aligned} \langle W_2 \gamma_x + \gamma_x W_1 \rangle &= 2m \langle \gamma_x \rangle \\ \langle \gamma_z \gamma_x \rangle &= \frac{m}{q} \langle \gamma_x \rangle \end{aligned} \right.$$

$$\left\{ \begin{aligned} \langle W_2 \gamma_z \rangle &= m \langle \gamma_z \rangle = 0 \\ p \langle \gamma_t \gamma_z \rangle + q \langle 1 \rangle &= 0 \end{aligned} \right.$$

In this way we can obtain all matrix elements in terms of $\langle 1 \rangle$, $\langle \gamma_x \rangle$, $\langle \gamma_x \gamma_y \rangle$. To obtain these we must consider the spin and the normalization.

Spin:

$$\Sigma_{xy} \Psi_1 = s_1 \Psi_1$$

$$\Sigma_{xy} \equiv +i\gamma_x\gamma_y = \sigma_z$$

$$\Sigma_{xy} \Psi_2 = s_2 \Psi_2$$

$$\langle \Sigma_{xy} \rangle = s_1 \langle 1 \rangle = s_2 \langle 1 \rangle$$

We have two cases to consider $s_1 = s_2$ and $s_1 = -s_2$

Case I $\quad p, \; q = \sqrt{m^2 + p^2}$

$$s_1 = -s_2$$

Normalization

$$\langle 1 \rangle = a p$$

$$\langle \gamma_x \rangle = \langle \gamma_y \rangle = \langle \gamma_z \rangle = 0$$

$$\langle \gamma_t \rangle = m a$$

$$\langle \gamma_x \gamma_y \rangle = -i s_1 a p$$

$$\langle \gamma_x \gamma_z \rangle = \langle \gamma_y \gamma_z \rangle = 0$$

$$\langle \gamma_t \gamma_z \rangle = q a$$

$$\langle \gamma_t \gamma_y \rangle = \langle \gamma_t \gamma_x \rangle = 0$$

$$\langle \gamma_x \gamma_y \gamma_z \rangle = \langle \gamma_t \gamma_z \gamma_x \rangle = \langle \gamma_t \gamma_z \gamma_y \rangle = 0$$

$$\langle \gamma_t \gamma_x \gamma_y \rangle = -i m s_1 a$$

$$\langle \gamma_{5_{txyz}} \rangle = i q s_1 a$$

Case II $\quad s_1 = -s_2$

Normalization $\quad \langle \gamma_x \rangle = q b$

$$\langle 1 \rangle = \langle \gamma_z \rangle = \langle \gamma_t \rangle = 0$$

$$\langle \gamma_y \rangle = -i s_1 b q$$

$$\langle \gamma_x \gamma_y \rangle = \langle \gamma_t \gamma_z \rangle = 0$$
$$\langle \gamma_z \gamma_x \rangle = m b$$
$$\langle \gamma_z \gamma_y \rangle = -i b s_1 m$$
$$\langle \gamma_t \gamma_x \rangle = \langle \gamma_t \gamma_y \rangle = 0$$
$$\langle \gamma_x \gamma_y \gamma_z \rangle = \langle \gamma_t \gamma_x \gamma_y \rangle = 0.$$
$$\langle \gamma_t \gamma_z \gamma_x \rangle = p b \quad \langle \gamma_t \gamma_z \gamma_y \rangle = -i b s_1 p$$
$$\langle \gamma_5 \rangle = 0$$

Coordinate Transformations

Use of invariants in transforming to center of gravity system ($\mathbf{p}_1 = -\mathbf{p}_2$)

$p_{2\mu} : E_2, \mathbf{p}_2$

$p_{1\mu} : E_1, \mathbf{p}_1$

Lab sys.

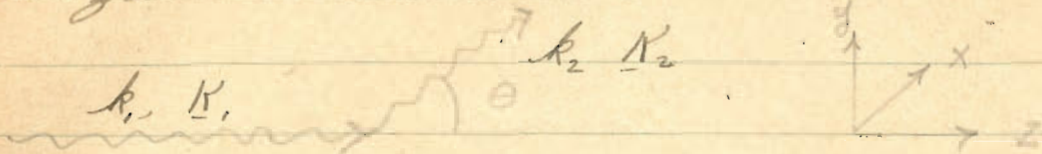
$p, -\mathbf{q}$

$p, \mathbf{q}$

c.g sys.

$$p_{1\mu} p_{1\mu} = m^2$$
$$p_{2\mu} p_{2\mu} = m^2$$
$$\frac{(p_{1\mu} + p_{2\mu})^2}{4} = \left(\frac{E_1 + E_2}{2}\right)^2 - \left(\frac{\mathbf{p}_1 + \mathbf{p}_2}{2}\right)^2 = p^2$$
$$\left(\frac{p_{1\mu} - p_{2\mu}}{2}\right)^2 = \left(\frac{E_1 - E_2}{2}\right)^2 - \left(\frac{\mathbf{p}_1 - \mathbf{p}_2}{2}\right)^2 = -q^2$$

Scattering of light: Choice of polarization directions.



Let e^a_u, e^b_u be unit vectors defining the two directions of polarization. We wish to choose these $\perp$ to each other and to the two beams of light. This csp to a tranformation to a system in which the light simply reverses its direction.
The conditions to be fullfilled are:

$$e^a_u e^a_u = e^b_u e^b_u = -1$$
$$e^a_u e^b_u = 0$$
$$e^a_u k^1_u = e^a_u k^2_u = e^b_u k^1_u = e^b_u k^2_u = 0$$
$$k^1_u k^1_u = k^2_u k^2_u = 0$$

		x	y	z	t
Let k^1_u	=	0	0	k_1	k_1
k^2_u	=	0	$\sin\theta\, k_2$	$\cos\theta\, k_2$	k_2
e^a_u	=	1	0	0	0
e^b_u	=	ξ	η	ζ	τ

From $e^a_u e^b_u = 0$; $\xi = 0$

From $e_\mu^b k_\mu^1 = 0$: $\tau = \zeta$

From $e_\mu^b k_\mu^2 = 0$

$$\tau k_2 - \tau k_2 \cos\theta - \eta k_2 \sin\theta = 0$$

$$\eta = \tau(1 - \cos\theta)/\sin\theta = \tau \tan\frac{\theta}{2}$$

From $e_\mu^b e_\mu^b = 1$

$$\tau^2 - \tau^2 - \eta^2 = -1$$

$$\eta = \pm 1$$

$$e_\mu^b = 0 \quad 1 \quad \cot\frac{\theta}{2} \quad \cot\frac{\theta}{2}$$

Spin: The spin is an axial vector. It is therefore a property of a given plane. In problems involving scattering it is convenient to refer the spin of the initial and final particle states to the same plane. This may be done in 4 dimensions in a manner exactly analogous to our treatment of the polarization directions for light scattering.

Let a_ν b_ν be unit vectors $\perp$ to each other and to the incident and final particle momenta

$$a_\mu^2 = -1$$
$$b_\mu^2 = -1$$
$$a_\nu b_\mu = a_\mu p_\mu = b_\mu p_\mu = 0$$

Define

$$\not{a} = a_\mu \gamma_\mu$$
$$\not{b} = b_\nu \gamma_\mu$$
$$\Sigma = i \not{a} \not{b}$$

then

$$[H, \Sigma] = 0$$
$$\Sigma \Psi = s \Psi$$

Σ is a spin operator which determines the spin $\perp$ to the plane of a_μ + b_μ.

Projection Operators

Theorem: If u_s^p form a complete orthonormal set and $a_s = (f^p, u_s^p)$

$$\sum_s |(f^p, u_s^p)|^2 = (f^p, f^p)$$

$$\sum_s (f^p, u_s^p)\overline{(g^p, u_s^p)} = (f^p, g^p)$$

Let u_s^p : $s = 1, 2, 3, 4.$ be the four solution of the D.E. for a free particle of momentum p.

$$\sum_s (u_a^{p*} A\, u_s^p)(u_s^{p*} B\, u_b^p) = (u_a^{p*} A B\, u_b^p)$$

An appropriate choice of A & B enables one to express sums over certain states in terms of the single matrix element on the right.

Examples:

To sum over states of + spin choose $B = 1$

$$A = \frac{\Sigma + 1}{2}$$

then since

$$A u^{p}_{s=+1} = \frac{1u^{p}_{+1} + 1u^{p}_{+1}}{2} = u^{p}_{+1}$$

and

$$A u^{p}_{s=-1} = \frac{-1u^{p}_{-1} + 1u^{p}_{-1}}{2} = 0$$

only those terms corr to $s = +1$ will appear in the sum.

To sum over states of $E = +E_p$ choose $B = 1$

$$A = \frac{H^{p} + E_p}{2E_p} \quad : H^{p} = \alpha \cdot p + \beta m$$

We may use these operators to compute $(u_1^* u_2)$ where:

Case I $\quad u_1 \; : \; +E_p^1 \, , \; s = +1$

$\qquad\quad u_2 \; : \; +E_p^2 \, , \; s = +1$

$$\sum_s \left(u_1^* \frac{H + E_p^2}{2E_p^2} \cdot \frac{\Sigma + 1}{2} u_s^{p_2}\right)\left(u_s^{p_2 *} u_1\right)$$

$$= (u_1^* u_2)(u_2^* u_1) = \left\{ u_1^* \frac{H + E_p^2}{2E_p^2} \cdot \frac{\Sigma + 1}{2} u_1 \right\}$$

$$= \left\{ u_1^* \frac{\alpha \cdot p_2 + \beta m + E_p^2}{2 E_p^2} u_1 \right\}$$

From table of matrix elements

$$= \frac{p_1 \cdot p_2}{2E^1E^2} + \frac{m^2}{2E^1E^2} + \frac{1}{2}$$

In C M system

$$p_1 = -p_2 = q$$

$$E_p^1 = E_p^2 = p$$

$$|u_2^* u_1|^2 = \frac{-q^2}{2p^2} + \frac{m^2}{2p^2} + \frac{p^2}{2p^2} = \frac{m^2}{p^2}$$

$$(u_2^* u_1) = m/p$$

Case II $\quad u_1 : E_p^1 \quad s = +1$

$\quad\quad u_2 : E_p^2 \quad s = -1$

$$\langle \tilde{u}_2 \gamma_x u_1 \rangle = (u_2^* \alpha_x u_1)$$

$$\sum_s \left(u_1^* \alpha_x \frac{H + E_p^2}{2E_p^2} \cdot \frac{\Sigma - 1}{-2} u_s\right)(u_s^* \alpha_x u_1)$$

$$= |(u_2^* \alpha_x u_1)|^2$$

$$= \left(u_1^* \alpha_x \frac{H + E_p^2}{2E_p} \cdot \frac{\Sigma - 1}{-2} \alpha_x u_1\right) \qquad \begin{array}{l} p = p_2 \\ \Sigma = -i\alpha_x\alpha_y \end{array}$$

$$= \left(u_1^* \alpha_x \frac{H + E_p^2}{2E_p} \alpha_x u_1\right)$$

$$= \left(u_1^* \frac{-\alpha \cdot p_2 + 2p_x - \beta m + E_p^2}{2E_p} u_1 \right) = \frac{g^2}{p^2}$$

$(u_2^* \alpha_x u_1) = g/p$ In C.M. system

Notation:

$$O_p^{\pm} \text{ or } \Lambda_p^{\pm} \equiv \frac{H_p^{\pm} + E_p}{2E_p}$$

These operators may be applied only to states of known momentum, p.

Transition Probability associated with a change A in an operator (coordinate)

$$\equiv |(u_1^* A u_2)|^2$$

In general the experimenter wants to know the prob of transition from a state $E_1 p_1$ to one $E_2 p_2$. He is not interested in the spin orientation of the states. We must therefore sum over the spins of the initial and final states.

$$\sum_{\substack{s_1\\+E_1}} \sum_{s_2} (u_{s_1}^{*} A \Lambda_{p_2}^{+} u_{s_2})(u_{s_2}^{*} A u_{s_1})$$

$$= \sum_{\substack{s_1\\+E_1}} (u_{s_1}^{*} A \cdot \Lambda_{p_2}^{+} A u_{s_1})$$

$$= \sum_{s_1} (u_{s_1}^{*} A \Lambda_{p_2}^{+} A \Lambda_{p_1}^{+} u_{s_1})$$

$$= \text{spur } A \Lambda_{p_2}^{+} A \Lambda_{p_1}^{+}$$

But spur (1) = 4 all other spurs = 0

$$\text{spur } (\alpha \cdot p_1)(\alpha \cdot p_2) = 4 p_1 \cdot p_2$$

$$\text{spur } (\alpha \cdot p_1)(\alpha \cdot p_2)(\alpha \cdot p_3) = 0$$

$$\text{spur } (\alpha \cdot p_1)(\alpha \cdot p_2)(\alpha \cdot p_3)(\alpha \cdot p_4)$$

$$= 4(p_1 \cdot p_2)(p_3 \cdot p_4) - 4(p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_4)(p_2 \cdot p_3)$$

Lemma: spur AB = spur BA

The last spur may be computed by pushing the first factor thru to the end and computing the residue from each commutation of factors.

Problem:

Show that for constant B
$\sigma \cdot \pi$ is a constant of the motion

$$\frac{d}{dt}(\sigma\cdot\pi) = \frac{i}{\hbar}(H\sigma\cdot\pi - \sigma\cdot\pi H) + \left(\frac{\partial(\sigma\cdot\pi)}{\partial t} = 0\right)$$

$$(\alpha\cdot\pi + \beta m)\sigma\cdot\pi - \sigma\cdot\pi(\alpha\cdot\pi + \beta m)$$

$$[\beta, \sigma\cdot\pi] = 0$$

$$[\alpha\cdot\pi, \sigma\cdot\pi] = 0$$

hence:

$$(\sigma\cdot\pi)\Psi = N\Psi$$

Term Problem:

Consider a free electron of momentum p and energy E_0. Let

$$A_\mu = e_{1\mu} a_1 (e^{ik_\mu^1 x_\mu} + e^{-ik_\mu^1 x_\mu}) + e_{2\mu} a_2 (e^{ik_\mu^2 x_\mu} + e^{-ik_\mu^2 x_\mu})$$

be a perturbing potential.

o: Show the electron makes transitions

$$E_0 + k_1 - k_2 = E_f$$

only if $p_f = p_0 + \underline{k}_1 - \underline{k}_2$

1: Find an expression for transition prob / sec in terms of matrix elements and sums on intermediate states. (corrected for induced Compton effect)
2: Find Compton x-section using Einstein A & B coefficients.
3: Express sums on intermediate states using projection operators.
4: Show result is same by hole theory.
5: Use coordinate system in which electron reverses its direction of motion. Calc. amplitude (matrix elements) for various polarizations and spins.
6: Find total x section for unpolarized γ-rays.
7: Calc. the total x-section in a coordinate system in which electron is initially at rest. (Klein Nishina)

8: Find from hole theory the probabilities for two quanta to create an electron pair.

a) Write down expression

b) Compute matrix elements

c) Compute total x-section

d) What is the life time of a positron in a gas of electrons of density N/cm^3 (This is not lifetime in matter)

Negative Energy solutions

Let p = momentum of an electron.
We have solutions of the form:

$$\psi^+ = u\, e^{-i(E_p t - \underline{p}\cdot\underline{x})}$$
$$\psi^- = u\, e^{-i(-E_p t - \underline{p}\cdot\underline{x})}.$$

Consider a wave packet representing a particle with momentum $\sim p_x^\circ$:

$$\psi(x\,t) = \int \phi(p_x)\, e^{-i(E(p_x)t - p_x x)}\, dp_x$$

Expand the exponent about p°

$$= e^{-i(E^\circ t - p^\circ x)} \int \phi(p)\, e^{-i(p - p^\circ)(\frac{\partial E}{\partial p}t - x)}\, dp.$$

The contribution of the integral will be largest for $(\frac{\partial E}{\partial p} t - x) = 0$. If the particle starts from $x = 0$ at $t = 0$, then at Δt it will be at Δx where:

$$\frac{\partial E}{\partial p}\Delta t - \Delta x = 0$$
$$\frac{\Delta x}{\Delta t} = V = \frac{\partial E}{\partial p}$$

This equation defines the velocity.
(Group velocity)

For $E = +E_p$

$$v^+ = \frac{\partial E_p}{\partial p} = p/E_p$$

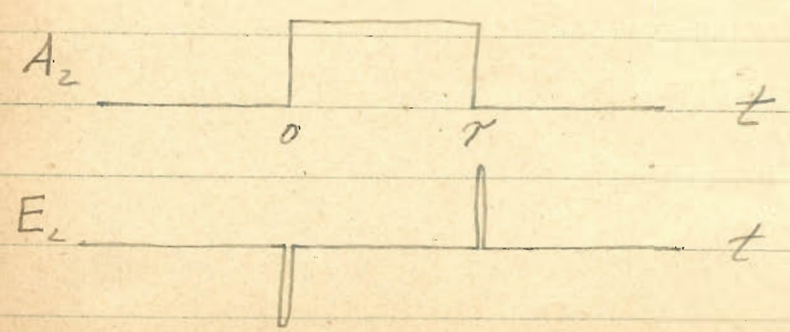
For $E = -E_p$

$$v^- = -\frac{\partial E_p}{\partial p} = -p/E_p$$

<u>Transition to Negative Energy State</u>

We shall now show by means of a simple example that it is possible to cause an electron in a positive energy state to jump into a negative energy state.

Let $A_z(t)$ be a vector potential constant in space but varying in time as shown below



We have:

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = (\alpha_2 A_2)\psi + \beta m \psi$$

$$-\frac{\hbar}{i}\frac{\partial \psi_a}{\partial t} = \sigma_2 A \psi_b + m\psi_a$$
$$-\frac{\hbar}{i}\frac{\partial \psi_b}{\partial t} = \sigma_2 A \psi_a - m\psi_b$$

Assume $\sigma_2 \psi_a = \psi_a$, $\sigma_2 \psi_b = \psi_b$
at $t = 0$

$$\psi_a(0) = 1 \quad : \quad -\frac{\hbar}{i}\frac{\partial \psi_a}{\partial t} = +E_p \psi_a$$
$$\psi_b(0) = 0 \quad : \quad -\frac{\hbar}{i}\frac{\partial \psi_b}{\partial t} = -E_p \psi_b$$

$0 < t < \tau$ assume solution

$$\psi_a(t) = u_a e^{-\frac{i}{\hbar}E t}$$
$$\psi_b(t) = u_b e^{-\frac{i}{\hbar}E t}$$

$$E u_a = A u_b + m u_a$$
$$-E u_b = A u_a - m u_b$$
$$E^2 = A^2 + m^2$$

$$\psi_b = c\left(e^{+iE_A t} - e^{-iE_A t}\right)$$
$$\psi_a = \frac{c}{A}\left\{(E_A + m)e^{iE_A t} - (-E_A + m)e^{-iE_A t}\right\}$$

$$\psi_a(0) = 1 \ : \ c = \frac{A}{2E_A}$$

At $t = \tau$

$$\psi_b = \frac{iA}{E_A} \sin E_A t$$
$$\psi_a = \cos E_A t + \frac{im}{E_A} \sin E_A t$$

Prob of being in neg E state at $t = \tau$

$$|\psi_b|^2 = \frac{A^2}{E_A^2} \sin^2 E_A t.$$

Symmetry of positive & negative Energy Solutions of D.E.

We shall now show that changing the sign of the charge in the D.E. has the effect of turning the positive energy solutions into negative energy solutions and vice-versa. We choose a representation in which the α's are real and β is imaginary. For example

$$\alpha_x' = \alpha_x \qquad \beta' = \alpha_y$$
$$\alpha_y' = \beta$$
$$\alpha_z' = \alpha_z$$

(The transformation which does this is $S = \frac{1}{\sqrt{2}}(1+\beta\alpha_y)$)

In this representation.

$$\frac{\hbar}{i}\frac{\partial \psi^*}{\partial t} = (\alpha^* \cdot \pi^* + \beta^* mc + e\phi)\psi^*$$
$$= (\alpha \cdot \pi^* - \beta mc + e\phi)\psi^*$$

If we change $e \to -e$

$$-\frac{\hbar}{i}\frac{\partial \psi^*}{\partial t} = (\alpha \cdot \pi + \beta mc + e\phi)\psi^*$$

$$\psi = \phi e^{-iEt}$$
$$\psi^* = \phi^* e^{+iEt}$$

ψ^* is known as the charge conjugate solution. The neg E solutions for ϕ csp to the pos energy solutions for ϕ^*.

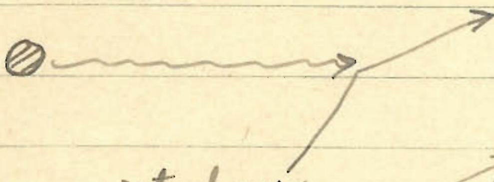
Hole Theory

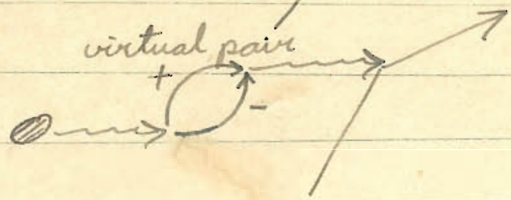
The negative energy states are a problem. Why don't all electrons disappear into them? One way of dealing with them is to postulate that they are all filled. This leads to obvious infinities as well as more subtle difficults such as polarization of the vacuum, but we will overlook these and see what useful information we can get.

Pair production: Suppose a γ ray knocks a neg electron into a positive energy state. This leaves behind a "hole" in the sea of neg E electrons. This hole appears to us as a positive electron. We have produced a "pair". To conserve energy and momentum 2 γ's are required to produce a pair.

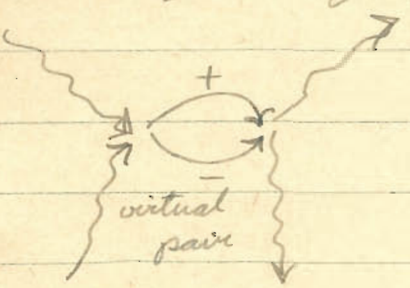
Virtual Processes

Suppose an electron going along decides to emit a light quanta. Having done so it finds that it has not conserved energy and momentum and so reabsorbs the quanta. Such a process is called <u>virtual</u>. The electron need not absorb the same quantum it emitted. It may instead choose to absorb a different quanta and be scattered. This is known as a <u>second order process</u> i.e. one in which the system goes thru an intermediate state in which energy need not be conserved. As another example of a virtual process consider the scattering of an electron by a nucleus. This may occur as pictured below:



or 

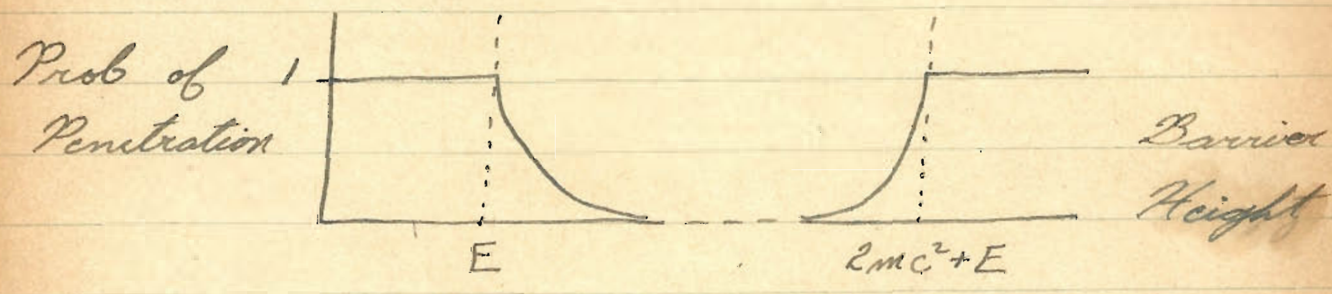
Scatter of light by light:



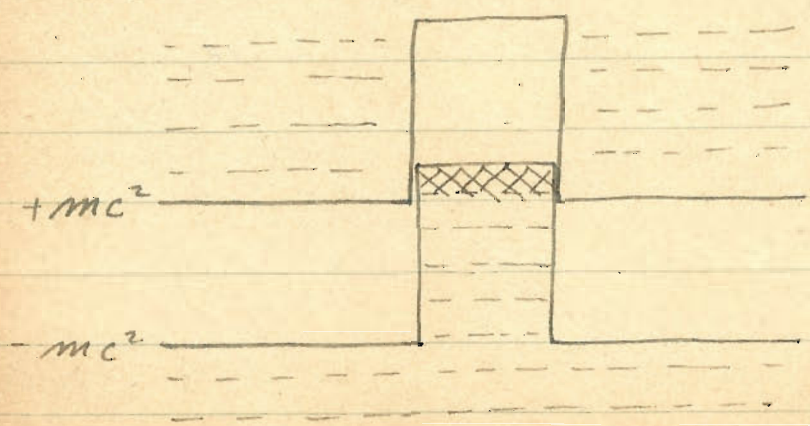
In general the higher order processes represent only small corrections to the lower order processes. It should be emphasized that it is only necessary to conserve E + p between initial and final state.

Klein Paradox

Klein calculated the probability that an electron of energy E would penetrate a square potential barrier. His results are illustrated by the graph below



The interesting point is that the probability of penetration rises to 1 again if the barrier becomes sufficiently high. We may explain this with the aid of the picture below:



Application of the potential causes the filled negative energy levels to stick up above the empty positive energy level bands. The negative energy electrons leak out leaving behind a region of "holes" (positrons) indicated by the cross hatching. Now if an electron comes along and strikes the barrier it may annihilate a positron and then leak out on the other side of the barrier. This same picture in reverse (i.e. upside down) is the basis for arguments which limit the size of Z on nuclei. If Z is sufficiently large, the potential in the region of the nucleus will fall below $-mc^2$ and electrons will accumulate about the nucleus limiting the effective Z which is measured.

<u>Quantum Electrodynamics</u>

In order to develope a Q.E. our first task is to put Maxwells equations into Hamiltonian form. We start with the Lagrangian for the E.M. field.

$$L = \frac{1}{8\pi}\int (E^2 - B^2)\,d\tau \qquad \text{Field}$$

$$+ \int j(\underline{x})\cdot A(\underline{x})\,d\tau - \int \rho(\underline{x})\,\phi(\underline{x})\,d\tau \qquad \begin{array}{l}\text{Interaction}\\ \text{of particles}\\ \text{\& field}\end{array}$$

$$+ \sum_i m_i \sqrt{1 - v_i^2} \qquad \text{Particles}$$

where
$$j(\underline{x}) = e_i v_i\, \delta(\underline{x}_i - \underline{x})$$
$$\rho(\underline{x}) = e_i\, \delta(\underline{x}_i - \underline{x})$$

We shall leave out the particle part.

$$= \frac{1}{8\pi}\int \left\{ \left[\nabla\phi + \frac{1}{c}\frac{\partial A}{\partial t}\right]^2 - (\nabla \times A)^2 \right\} d\tau$$

$$+ \int (j\cdot A - \rho\,\phi)\,d\tau$$

In writing the Lagrangian in this form we have assumed the

Maxwell equations

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

We can show that the Lagrangian is consistent with the M equations by using the least action principle to derive the other two.

$$\delta \int L\,dt = 0$$

A and ϕ are the coordinates to be varied. It works. Varying A gives $-\frac{1}{c}\frac{\partial E}{\partial t} + 4\pi j = \nabla \times B$. Varying ϕ gives $\nabla \cdot E = 4\pi\rho$

From the Lagrangian we can get the conjugate momenta.

$$\Pi_{A_x} = \frac{\partial L}{\partial \dot{A}_x}$$

Getting π involves a little trickery. Actually π is the momentum per unit volume. To perform the differentiation of the integral we think of it as being a sum over small volumes ϵ^3. If we consider only the volume element at the point $\underline{x}$, we obtain

$$\Pi_{A_x(\underline{x})} = \frac{\partial L}{\partial \dot{A}_x(\underline{x})} = \frac{1}{4\pi c} E_x(\underline{x})$$

$$\Pi_{\phi(\underline{x})} = \frac{\partial L}{\partial \dot{\phi}(\underline{x})} = 0$$

This last equation csp to the existance of an ignorable coordinate. This will prove a nuisance in what is to follow.

We introduce the Gauge Equation,

$$\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot A = 0$$

This implies a restriction on the variations we make in the Lagrangian. By means of Lagrange multipliers it is possible to define a new Lagrangian in which the variations of A & ϕ may be made in an arbitrary way.

$$\mathcal{L} = \frac{1}{8\pi}\iint\Big\{\Big(-\frac{1}{c^2}\Big(\frac{\partial\phi}{\partial t}\Big)^2 + (\nabla\phi)^2\Big)$$

$$+\sum_{i=xyz}\Big(\frac{1}{c^2}\Big(\frac{\partial A_i}{\partial t}\Big)^2 - (\nabla A_i)^2\Big)\Big\}\,d\tau\,.$$

$$+\int(j\cdot A - \rho\phi)\,d\tau$$

Vary ϕ to get $\square^2\phi = 4\pi\rho/c$

$$\square^2 A = 4\pi j$$

Using the $\mathcal{L}$ one obtains for the momenta:

$$\Pi_{A_x(\underline{x})} = \frac{1}{4\pi c^2}\frac{\partial A_x(\underline{x})}{\partial t}$$

$$\Pi_{\phi(\underline{x})} = -\frac{1}{4\pi c^2}\frac{\partial\phi(\underline{x})}{\partial t}$$

We can now write out the Hamiltonian:

$$H = \sum p\dot{q} - \mathcal{L}$$

$$= \int\Pi_\phi\frac{\partial\phi}{\partial t}\,d\tau + \int\Pi_{A_x}\frac{\partial A_x}{\partial t}\,d\tau + {}_y + {}_z$$

$$-\mathcal{L}$$

$$= 2\pi \int \{ -(\Pi_\phi^2 c^2 + \frac{(\nabla\phi)^2}{8\pi}) + \sum_i (\Pi_{A_i}^2 c^2 + \frac{(\nabla A_i)^2}{8\pi}) \} d\tau$$

$$- \int [j \cdot A - \rho\phi] \, d\tau$$

$$+ H_{\text{particle}}$$

Check: $\frac{\partial H}{\partial \Pi_{A_x}} = \frac{\partial A_x}{\partial t} = 4\pi c^2 \Pi_{A_x}$

$$\frac{\partial H}{\partial \Pi_\phi} = \frac{\partial \phi}{\partial t} = -4\pi c^2 \Pi_\phi$$

$$-\frac{\partial H}{\partial A_x} = \frac{\partial \Pi_{A_x}}{\partial t} = -\frac{1}{4\pi}\nabla^2 A_x + j_x$$

To get Q.M. we need only introduce the appropriate commutation relations

$$[\Pi_{A_i(\underline{x})}, A_j(\underline{x}')] = \delta_{ij} \frac{\hbar}{i} \delta(\underline{x} - \underline{x}')$$

In principle, we could now apply our H to a super duper wave function which depended on infinitely many coordinates. It turns out this is impractical and that it is better to introduce new coordinates.

Nov. 6, 1948

Introduce the coordinates

$$q^c_{\underline{k}x} = \int \cos(\underline{k}\cdot\underline{x})\, A_x(\underline{x})\, d\underline{x}$$

Then $A_x(\underline{x}) = \sqrt{8\pi c^2} \sum_{\underline{k}} \{ q^c_{\underline{k}x} \cos(\underline{k}\cdot\underline{x}) + q^s_{\underline{k}x} \sin(\underline{k}\cdot\underline{x}) \}$

where $\sum_{\underline{k}} \to \int \frac{d\underline{k}}{(2\pi)^3}$

Similarly $\phi(x) = \sqrt{8\pi c^2} \sum_{\underline{k}} \{ r^c_{\underline{k}} \cos(\underline{k}\cdot\underline{x}) + r^s_{\underline{k}} \sin(\underline{k}\cdot\underline{x}) \}$

The Lagrangian becomes

$$L = \int \frac{d\underline{k}}{(2\pi)^3} \{ -\tfrac{1}{2} \dot{r}^{sc\,2}_{\underline{k}} + \tfrac{1}{2} \underline{k}^2 r^{sc\,2}_{\underline{k}} \}$$

$$+ \{ \tfrac{1}{2} \dot{q}^{sc\,2}_{\underline{k}} - \tfrac{1}{2} \underline{k}^2 q^{sc\,2}_{\underline{k}} \}$$

$$+ \sqrt{4\pi c^2} \int \frac{d\underline{k}}{(2\pi)^3} \sum_j \{ e_j (V_j \cdot q^c_{\underline{k}} - r^c_{\underline{k}}) \cos \underline{k}\cdot\underline{x}_j$$

$$+ e_j (V_j \cdot q^s_{\underline{k}} - r^s_{\underline{k}}) \sin \underline{k}\cdot\underline{x}_j \} + L_{material}$$

Here the q's are vectors and the shorthand

$$r^{2^{sc}} = r^{2^{s}} + r^{2^{c}}$$

is used.

The momenta are

$$s_{\underline{k}}^{sc} = \frac{\partial L}{\partial \dot{r}_{\underline{k}}^{sc}} = -\dot{r}_{\underline{k}}^{sc}$$

$$p_{x\underline{k}}^{sc} = \frac{\partial L}{\partial \dot{q}_{x\underline{k}}^{sc}} = \dot{q}_{x\underline{k}}^{sc}$$

The Hamiltonian

$$H = H_{rad} + H_{int} + H_{mat}$$

$$= \int \frac{d\underline{k}}{(2\pi)^3} \left\{ \frac{1}{2}\left(p_{\underline{k}}^{2^{sc}} + \underline{k}^2 q_{\underline{k}}^{2^{sc}}\right) - \frac{1}{2}\left(s_{\underline{k}}^{2^{sc}} + \underline{k}^2 r_{\underline{k}}^{2^{sc}}\right) \right\}$$

$$+ \sqrt{4\pi e^2} \int \frac{d\underline{k}}{(2\pi)^3} \sum_j e_j \left\{ \left(r_{\underline{k}}^{c} - \underline{\alpha}_j \cdot q_{\underline{k}}^{c}\right) \cos \underline{k}\cdot\underline{x}_j \right.$$

$$\left. + \left(r_{\underline{k}}^{s} - \underline{\alpha}_j \cdot q_{\underline{k}}^{s}\right) \sin \underline{k}\cdot\underline{x}_j \right\}$$

$$+ \sum_j \left\{ \alpha_j \cdot \frac{\hbar}{i} \nabla_j + \beta_j m_j \right\}$$

Here the α_j and β_j are $4^N \times 4^N$ (where N = # of particles) matrices obtained by appropriate iteration of

α & β. For example for 2 particles, a & b

$$\psi = \begin{array}{cc} a & b \\ \hline 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 2 & 1 \\ 2 & 2 \\ \vdots & \vdots \end{array}$$

11 12 13 .... 41 42 43 44

α_{xa} 11 1

12 $\quad 21|\overline{11}^{\,31}$. 1

13 $\quad 22|\overline{11}^{\,32}$ 1

14 $\quad 31|\overline{11}^{\,21}$ 1

$\vdots \quad 41|\overline{11}^{\,11} \quad 32|\overline{11}^{\,22}$ . . . .

$42|\overline{11}^{\,12}$

To develope this matrix of α_{xa} for particle a, one considers the 4 4 × 4 matrices formed by letting particle b be in its 4 given states and fills in each of these as α_x.

For the p's & q's, r & s

$$[p^c_{xk}\ p^c_{xk}] = \frac{1}{i}$$

Canonical transformation to running waves:

$$P_{\underline{k}} = p^c_{\underline{k}} + k q^s_{\underline{k}} \qquad P_{-\underline{k}} = p^c_{\underline{k}} - k q^s_{\underline{k}}$$

$$k Q_{\underline{k}} = -p^s_{\underline{k}} + k q^c_{\underline{k}} \qquad k Q_{-\underline{k}} = p^s_{\underline{k}} + k q^c_{\underline{k}}$$

We have:

$$H_{rad.} = \int \frac{d\underline{k}}{(2\pi)^3}\left\{\frac{1}{2}(P_k^2 + k^2 Q_k^2) - \frac{1}{2}(S_k^2 + k^2 R_k^2)\right\}$$

$$H_{int} = -\sqrt{4\pi c^2}\int \frac{d\underline{k}}{(2\pi)^3}\sum_{j}\left\{[e_j(\alpha\cdot q^c_k - r^c_k)e^{i\underline{k}\cdot\underline{x}} + \text{conj.}]\frac{1}{2}\right.$$

$$\left. + [e_j(\alpha\cdot q^s_k - r^s_k)e^{i\underline{k}\cdot\underline{x}} - \text{conj}]\frac{1}{2i}\right.$$

$$Q_{n,n+1} = \sqrt{\frac{\hbar(n+1)}{2k}}\ e^{-ikt}$$

$$Q_{n,n-1} = \sqrt{\frac{\hbar(n)}{2k}}\ e^{ikt}$$

We can separate absorbtion & emission by a further transformation.

$$q = Q + \frac{iP}{k}, \quad q^* = Q - \frac{iP}{k}$$

$$q_{n-1,n} = \sqrt{\frac{\hbar n}{2k}}\, e^{-ikt} \qquad \text{absorbtion}$$

$$q^*_{n+1,n} = \sqrt{\frac{\hbar(n+1)}{2k}}\, e^{+ikt} \qquad \text{emission}$$

$$[q\, q^*] = \frac{1}{2k}$$

If we subtract off the zero-point energy we may write:

$$H_{rad} = \int \frac{dk}{(2\pi)^3}\left[k^2[(q_k q_k^* + q_k^* q_k) - (r_k r_k^* + r_k^* r_k)] - \tfrac{1}{2}k\right]$$

$$= \int \frac{dk}{(2\pi)^3}\left[2k^2[q_k^* q_k - r_k^* r_k]\right]$$

where we have subtracted of the zero pt. energy $\frac{1}{2}k$.

Consider:

$$H = \int \frac{dk}{(2\pi)^3} \left\{ \tfrac{1}{2}\left(p_k^{sc^2} + k^2 q_k^{sc^2}\right) - \tfrac{1}{2}\left(s_k^{sc^2} + k^2 r_k^{sc^2}\right) \right\}$$

$$+ \sqrt{4\pi c^2} \int \frac{dk}{(2\pi)^3} \sum_j e_j \left\{ (r_k^c - \alpha_j \cdot q_k^c) \cos k \cdot x_j \right.$$

$$\left. + (r_k^s - \alpha_j \cdot q_k^s) \sin k \cdot x_j \right\}$$

$$+ H_{mat.}$$

$$\frac{\partial H}{\partial p} = \dot{q} \quad : \quad \frac{\partial H}{\partial q} = -\dot{p}$$

$$\dot{q}_k^c = p_k^c$$

1) $\ddot{q}_k^c = \dot{p}_k^c = -\left[k^2 q_k^c - \sqrt{} \sum_j e_j \alpha_j \cos k \cdot x_j \right]$

$$\dot{r}_k^c = -s_k^c$$

2) $\ddot{r}_k^c = \dot{s}_k^c = \left[k^2 r_k^c - \sqrt{} \sum_j e_j \cos k \cdot x_j \right]$

1) & 2) are the Maxwell wave equations for A & ϕ.

In addition to the Maxwell equations we must also satisfy the Gauge condition

$$\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot A = 0$$

In terms of our new coord.

$$s_k^c - k \cdot q_k^s = 0$$

This must be satisfied at all times. Hence at a given time, it and all its derivatives must equal zero.

Consider:

$$\frac{d^2}{dt^2}(s_k^c - k \cdot q_k^s) = -k^2(s_k^c - k \cdot q_k^s)$$

Hence if $(s_k^c - k \cdot q_k^s)$ and $(s_k^c - k \cdot q_k^s)'$ $= 0$ at $t = 0$ then everything O.K.

We can use this condition to eliminate the longitudinal and scalar waves:

Choose as coord

$$r_k^c, \quad \bar{q}_k^c = \frac{k}{|k|} \cdot q_k^c$$

$$q_{k_1}^c \perp \bar{q}_k^c, \quad q_{k_2}^c \perp \bar{q}_k^c, q_{k_1}^c$$

From the Gauge cond and $s_k^c = \frac{1}{i}\frac{\partial}{\partial r_k^c}$

$$\Psi = e^{i\sum_k k\bar{q}_k^s r_k^c}\,\Psi''(\text{ind of } r)$$

From $\frac{d}{dt}\{\text{Gauge cond.}\} = 0$

$$\Psi = e^{\sum_k \{r_k^c k \bar{q}_k^s - \sum \frac{1}{k} e_j (\cos k\cdot x_j)\bar{q}_k^s\}}\,\Psi'''(\text{ind of } \bar{q}_k^s)$$

$\therefore$

$$\Psi = \Phi \exp i \sum_k \Big\{\bar{q}_k^s\Big(kr_k^c - \frac{1}{k}\sum_j e_j \cos k\cdot x_j\Big) + \bar{q}_k^c\Big(kr_k^s - \frac{1}{k}\sum_j e_j \sin k\cdot x_j\Big)\Big\}$$

where Φ is ind. of r & $\bar{q}$

$$\Psi = \Phi e^{iS}$$

$$H\Psi = -\frac{1}{i}\frac{\partial\Psi}{\partial t} \quad \therefore \quad H\Phi e^{iS} = -\frac{1}{i}\frac{\partial \Phi e^{iS}}{\partial t}$$

$$H_F\Phi = e^{-iS} H e^{iS}\Phi = -\frac{1}{i}\frac{\partial}{\partial t}\Phi(q_{k1}, q_{k2})$$

$$H_F = \int \frac{dk}{8\pi^3}\left\{ \frac{1}{2}(p^s_{1k})^2 + \frac{1}{2}k^2(q^s_{1k})^2 + \frac{1}{2}(p^c_{1k})^2 + \frac{1}{2}k^2(q^c_{1k})^2 + \text{ditto } 2 \right\}$$

$$+ \sqrt{} \int \frac{dk}{8\pi^3}\left\{ \sum e_j \left(\alpha'_j q^c_{1k} \cos(k\cdot x_j) + \alpha'_j q^s_{1k} \sin(k\cdot x_j)\right) + \text{ditto } 2 \right\}$$

$$+ H_{mat} + H_{coulomb}$$

$$H_{coul} = \frac{1}{2}\sum_{ij} e_i e_j \int \frac{dk}{(2\pi)^3}\left\{ \frac{\cos k\cdot x_i \cos k\cdot x_j}{k^2} + \frac{\sin k\cdot x_i \sin k\cdot x_j}{k^2} \right\}$$

$$= \frac{1}{2}\sum_{ij} \frac{e_i e_j}{|r_{ij}|}$$

<u>Hamiltonian</u> <u>in</u> <u>terms</u> <u>of</u> <u>q + q^*</u>

correct with factors (?)

$$H_{rad} = \int \frac{dk}{8\pi^3} 2k^2 \left(q^*_{1k} q_{1k} + q^*_{2k} q_{2k} \right)$$

$$H_{int} = \sqrt{4\pi c^2} \int \frac{dk}{8\pi^3} \sum_j \left(e_j \alpha'_j q_{1k} e^{ik\cdot x_j} + e_j \alpha' q^*_{1k} e^{-ik\cdot x_j} + \text{ditto } 2 \right)$$

$$H_{atoms} = H_{mat} + H_{coul}$$

$$= \sum_j \left\{ \frac{1}{i} \alpha_j \cdot \nabla_j + \beta_j m_j \right\} + \frac{1}{2} \sum_{ij} \frac{e_i e_j}{|r_{ij}|}$$

The last term here contains an infinity. This is wrong. It turns out, however, that if one computes everything in the first order in which it does not vanish things are alright. The corrections in most cases are infinite. This matter is currently being straightened out.

Emission of Light from an Atom

Initial state: Atom in level E_a & all osc. of field in ground state.

Final state: Atom in lower level E_b & all osc. in ground state except one in state k.

It is of course impossible to have all osc. in ground state but one can probably demonstrate that having the atom in a light tight box will not introduce any appreciable error.

From time-dependent perturbation theory

$$\text{Prob } i \to f = \frac{2\pi}{\hbar} \rho(E_f) |M_{if}|^2 : E_i = E_f$$

1^{st} order $\quad M_{if} = H_{if}$

2^{nd} order $\quad M_{if} = \sum_I \frac{H_{fI} H_{Ii}}{E_I - E_i} \quad$ etc.

Illustrative Prob.

i : atom in E_a

f : atom in E_b + one qu. $\underline{k}$, pol. $\underline{e}_1$

$$E_a = E_b + k$$

$$\Psi_i = \Psi_a(x_{atom}) \sum \phi_0(q_{i,1})\phi_0(q_{i,2})$$

$$\Psi_f = \Psi_b(x_{atom}) \phi_1(q_{k,1}) \phi_0(q_{k,2}) \sum_{i \neq k} \phi_0(q_{i1})\phi_0(q_{i2})$$

When the matrix element of H_{int} is taken between Ψ_f & Ψ_i the only surviving term will be:

$$\int e \Psi_f^* \{\alpha' q_{k_1} e^{i\underline{k}\cdot\underline{x}} + \alpha' q_{k_1}^* e^{-\underline{k}\cdot\underline{x}}\} \Psi_i \, d\tau$$

The first term here will vanish since $(q)_{10} = 0$

$$H_{fi} = (q^*)_{10} \sqrt{4\pi c^2}\, e \int \phi_b^* [(\alpha \cdot e_1) e^{-ik\cdot x}] \phi_a \, dx$$

$$\rho(E) = \rho(k)\frac{dk}{dE} = \frac{k^2 dk\, d\Omega}{(2\pi)^3}\frac{1}{dE}$$

$$= \frac{k^2 d\Omega}{(2\pi)^3}$$

Prob / sec $= \frac{c^2 k\, d\Omega}{2\pi \hbar} e^2 \left| \alpha \cdot e_1 e^{-i\underline{k}\cdot\underline{x}} \right|^2_{ba}$

In non-relativistic limit

$$\alpha = \frac{v}{c} = \frac{p}{mc}$$

Dipole approx. $\lambda >> x_{atom}$

$$\underline{k} \cdot \underline{x} << 1$$

$$e^{-i\underline{k}\cdot\underline{x}} = 1$$

This approx assumes there is no phase change across the atom. Not valid for X-rays.

$$|p_1|_{ba} = i\, m \frac{\overbrace{E_b - E_a}^{ck\hbar}}{\hbar} |x_1|_{ba}$$

follows from $\frac{p}{m} = \frac{i}{\hbar}(Hx - xH)$

Prob / sec $= \frac{c k^3}{2\pi \hbar} e^2 |x_1|^2_{ba}\, d\Omega$

Total radiation prob. / sec

$$\sigma_{ab} = \frac{2}{3} \frac{c k^3}{2\pi \hbar} e^2\, x_{ba} \cdot x_{ba}$$

Classical formula energy / sec

$$= \frac{2}{3} \frac{e^2}{c^3} (\omega^2 x) \cdot (\omega^2 x)$$

If there are n quanta initially:
rate for emission = $(n+1)$ rate with 0 qu.
rate for absorb = n rate with one
qu. initially

$$H_{b\,n_{\underline{k},1},\ a\,n+1_{\underline{k},1}} = \sqrt{\frac{2\pi c^2 \hbar^2 (n+1)}{k}} \left(\sum_j e_j \alpha_j \cdot \underline{e}_1 e^{i\underline{k}\cdot\underline{x}} \right)_{ba}$$

$$H_{b\,n+1_{\underline{k},1},\ a\,n_{\underline{k},1}} = \sqrt{\frac{2\pi c^2 \hbar^2 (n+1)}{k}} \left(\sum_j e_j \alpha_j \cdot \underline{e}_1 e^{-i\underline{k}\cdot\underline{x}} \right)_{ba}$$

Problem: Calc. Prob of emission of 2 quanta. Compare with prob for one quantum.

Dispersion & Scattering

Initial: atom E_a, qu $\underline{k}_1$ $\underline{e}_1$
Final: atom E_b, qu $\underline{k}_2$ $\underline{e}_2$
where $E_a + k = E_b + k_2$

This is a second order process. i.e. one involving 2 quanta. It may occur in two ways

	I	II
H_{Ii}	abs $\underline{k}_1$, $E_a \to E_n$	em $\underline{k}_2$, $E_a \to E_n$
E_I =	E_n	$E_n + k_1 + k_2$
H_{fI}	em $\underline{k}_2$, $E_n \to E_b$	abs $\underline{k}_1$, $E_n \to E_b$

$$M_{fi} = \ldots \sum \left\{ \frac{(\alpha^2 e^{-i\underline{k}_2\cdot\underline{x}})_{bn} (\alpha^1 e^{i\underline{k}_1\cdot\underline{x}})_{na}}{E_a + k_1 - E_n} + \frac{(\alpha^1 e^{i\underline{k}_1\cdot\underline{x}})_{bn} (\alpha^2 e^{-i\underline{k}_2\cdot\underline{x}})_{na}}{E_a - E_n - k_2} \right\}$$

Problem: Show

$(\alpha e^{-ik\cdot x})_{fi}$ expressed in terms of large components of Ψ, Ψ_a, is

$$\int \Psi_{fa}^* \left\{ \frac{p}{2m} e^{-i\underline{k}\cdot\underline{x}} + e^{-i\underline{k}\cdot\underline{x}} \frac{p}{2m} \right.$$

$$\left. + \frac{1}{2mc^2} i(\sigma \times \underline{k}) e^{-i\underline{k}\cdot\underline{x}} \right\} \Psi_{ia}\, d\tau$$

In dipole approx $= \int \Psi_{fa}^* \frac{p}{m} \Psi_{ia}\, d\tau$

$$M_{fi} = \ldots \left\{ \sum_n \left[\frac{(p_2)_{bn}(p_1)_{na}}{E_a + k_1 - E_n} + \frac{(p_1)_{bn}(p_2)_{na}}{E_a - E_n - k_2} \right] \right.$$

$$\left. + \sum_{\text{neg } E} \left[\frac{(\alpha_2)_{bn}(\alpha_1)_{na}}{E_a + k_1 - E_n} + \frac{(\alpha_1)_{bn}(\alpha_2)_{na}}{E_a - E_n - k_2} \right] \right\}$$

With respect to the sum of neg E states one may argue that only states of low momentum, ie $E_n \approx -mc^2$, will contribute because states of high momentum (neg) will be orthogonal to the low momentum states of + E which are our initial and final states.

$$\sum_{neg\,E_n} \frac{(\alpha_1)_{bn}(\alpha_2)_{na} + (\alpha_2)_{bn}(\alpha_1)_{na}}{2mc^2}$$

$$\approx \sum_{all\,E} \frac{(\alpha_1)_{bn}(\alpha_2)_{na} + (\alpha_2)_{bn}(\alpha_1)_{na}}{2mc^2}$$

Since the numerator is small compared to $2mc^2$ for $+E$ states.

$$\sim \frac{2(e_1 \cdot e_2)_{ba}}{2mc^2} \sim \delta_{ba}(e_1 \cdot e_2)$$

Handling of Negative Energy States

Hole Theory Since in hole theory the negative energy states are filled, we must replace our old neg. E terms by new allowed terms:

(In the following, n is a neg. energy state)

H_{I}	em k_2 $E_n \to E_b$	abs k_1 $E_n \to E_b$
H_{II}	abs k_1 $E_a \to E_n$	em k_2 $E_a \to E_n$

$$M_{fi\,\mathrm{neg}} = \sum_n (-1)\left\{ \frac{(1)_{na}(2)_{bn}}{-E_b - k_2 - |E_n|} + \frac{(2)_{na}(1)_{bn}}{k_1 - E_b - |E_n|} \right\}$$

Remembering $E_a + k_1 = E_b + k_2$, we see this is the same result as that previously obtained.

The minus sign arises because in the process we have exchanged an electron in a neg E state with one in a pos E state. The wave function is antisymmetric for this exchange. In general, each interchange of a pair of electrons multiplies by (-1)

The process described above is alright unless $E_a = E_b$. If $E_a = E_b$ it is impossible because we would then have 2 electrons occupying the same state.

We consider for the moment a different problem: The scattering of light by a bare nucleus, potential.

$$H_{If} \qquad \overset{\text{abs } \underline{k}_1}{E_n \to E_m} \qquad\qquad \overset{\text{em } \underline{k}_2}{E_n \to E_m}$$

$$H_{fI} \qquad \overset{\text{em } \underline{k}_2}{E_m \to E_n} \qquad\qquad \overset{\text{abs } \underline{k}_1}{E_m \to E_n}$$

where $k_1 = k_2$

$$M_{fi}^{vac} = \ldots \sum_{\substack{m_+ \, n_-}} \left\{ \frac{(2)_{nm}(1)_{mn}}{-|E_n| - E_m + k_1} + \frac{(1)_{nm}(2)_{mn}}{-|E_n| - E_m - k_2} \right\}$$

Final result:

$$M_{fi}^{true} = M_{fi}^{atom} + M_{fi}^{vac} - \left(\begin{matrix} E_a = E_b \\ \text{term} \end{matrix} \right)$$

Scattering of two Electrons

1) Coulomb effect
2) Interaction of magnetic fields
3) Spin interaction

$$H' = \frac{e^2}{r_{12}} + H_{int.}$$

$$M_{fi} = H'_{fi} + \sum_I \frac{H_{iI} H_{If}}{E_i - E_I}$$

$$= H^{coul}_{fi} + \sum_I \frac{H_{int\,iI} H_{int\,If}}{E_i - E_I}$$

Coulomb interaction:

$$M^{coul}_{fi} = \int \psi^*_{f_2} \psi^*_{f_1} \frac{e^2}{r_{12}} \psi_{i_2} \psi_{i_1} \, d\tau_1 \, d\tau_2$$

$$= e^2 (u_f^* u_i)_2 (u_f^* u_i)_1 \int \frac{e^{-i(p_f - p_i)_2 \cdot r_2 - i(p_f - p_i)_1 \cdot r_1}}{r_{12}} \, d\tau_1 \, d\tau_2$$

$$= \quad '' \quad \int e^{-i[(p_f - p_i)_2 + (p_f - p_i)_1] \cdot r_2} \, d\tau_2 \int \frac{e^{-(p_f - p_i) \cdot r_{12}}}{r_{12}} \, d\tau_1$$

For non zero result:

$$(p_f - p_i)_2 = -(p_f - p_i)_1$$

Second integral may be integrated by using polar coord.

$$= \frac{4\pi}{q^2}$$

$$M_{fi}^{coul} = \ldots \frac{4\pi e^2}{q^2} (u_f u_i)_2 (u_f u_i)_1$$

<u>Interaction</u>

$$A. \qquad E_{i1} \xrightarrow{\text{em } \underline{k}} E_{f1} \qquad\qquad E_{i2} \xrightarrow{\text{abs } \underline{k}} E_{f2}$$

$$B. \qquad E_{i2} \xrightarrow{\text{em } \underline{k}} E_{f2} \qquad\qquad E_{i1} \xrightarrow{\text{abs } \underline{k}} E_{f1}$$

For A.
$$E_I = E_{f1} + E_{i2} + \hbar c k$$
$$E_i - E_I = E_{i1} - E_{f1} - \hbar c k = Q - k$$

$$H_{Ii} = \sqrt{\frac{2\pi \hbar c^2}{k}} \left(\sum_{j=1}^{2} \alpha_j \cdot e_a e^{-i\underline{k}\cdot \underline{r}_j} \right)_{Ii}$$

$$= \text{''} \int \psi_{f2}^* \psi_{f1}^* \left(\sum_1^2 \alpha_j \cdot e_a e^{-i\underline{k}\cdot r_j} \right) \psi_{i2} \psi_{i1} \, d\tau_1 \, d\tau_2$$

No contribution from α_2 because electron #1 makes

Integration over τ_2 yields one from normalization.

This leaves

$$H_{If} = \sqrt{\frac{2\pi\hbar c^2}{k}}\,(u_{f_1}\,\alpha_1\cdot e_a\,u_{i_1})\int e^{-i\underline{k}\cdot\underline{r}_1}\,e^{-i(\underline{p}_{f_1}-\underline{p}_{i_1})\cdot\underline{r}_1}\,d\tau_1$$

which vanishes unless

$$\underline{k} = -\,\underline{p}_{f_1} - \underline{p}_{i_1} = q$$

$$H_{Ii} = \sqrt{\frac{2\pi\hbar c^2}{k}}\,(u_{f_1}^*\,\alpha_1\cdot e_a\,u_{i_1})$$

Similarly

$$H_{fI} = \text{''} \quad (u_{f_2}^*\,\alpha_2\cdot e_a\,u_{i2})$$

$$M_{fi}^{\text{int}\,A} = \ldots\ \frac{1}{q}\,\frac{(u_{f_1}^*\,\alpha_1\cdot e_a\,u_{i_1})(u_{f_2}^*\,\alpha_2\cdot e_a\,u_{i2})}{Q-q}$$

For B everything is identical except

$$q = -k$$

$$M_{fi} = \ldots\ \frac{2\,(u_{f_2}^*\alpha_2\cdot e_a\,u_{i2})(u_{f_1}^*\alpha_1\cdot e_a\,u_{i2})}{Q^2-q^2}$$

but $$\sum_{\substack{\text{dir of}\\ \text{pol.}}}(A\cdot e_a)(B\cdot e_b) = A\cdot B - \frac{A\cdot q}{q}\cdot\frac{B\cdot q}{q}$$

$$= \frac{2}{Q^2-q^2}\left[(u_{f_2}^*\,\alpha_2\,u_{i2})\cdot(u_{f_1}^*\,\alpha_1\,u_{i1})\right]$$

$$-\ \frac{2}{Q^2-q^2}\cdot\frac{1}{q^2}\,(u_{f_2}^*(\alpha_2\cdot q)u_{i2})(u_{f_1}^*(\alpha_1\cdot q)u_{i1})$$

We want to compute:

$$(u_{f_2}^* (\alpha_2 \cdot p_{f_2} - \alpha_2 \cdot p_{i2}) u_{i2})$$
$$= (u_{f_2}^* (\alpha_2 \cdot p_{f_2} + \beta u - \alpha_2 \cdot p_{i2} - \beta u) u_{i2})$$
$$= (u_{f_2}^* (H_f - H_i) u_{i2})$$
$$= (E_{f2} - E_{i2})(u_f^* u_i)_2 \quad = Q (u_{f_2}^* u_{i2})$$

Hence for second term we get:

$$\frac{2}{Q^2-q^2} \frac{Q^2}{q^2} (u_f^* u_i)_2 (u_f^* u_i)_1$$

to this add the coulomb term

$$\frac{2}{q^2} (u_f^* u_i)_2 (u_f^* u_i)_1$$

to obtain

$$-\frac{2}{Q^2-q^2} (u_f^* u_i)_2 (u_f^* u_i)_1$$

For electron electron scattering

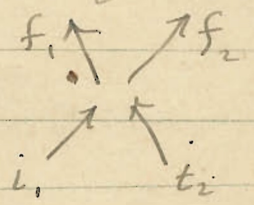
$$M_{fi} = \frac{2}{Q^2-q^2} \left[(u_{f_2}^* \alpha_2 u_{i2}) \cdot (u_{f_1} \alpha_1 \cdot u_{i1}) - (u_{f_2} u_{i2})(u_{f_1} u_{i1}) \right]$$

$$= \frac{2}{q_\mu q_\mu} (\tilde{u}_{f_2} \gamma_0 u_{i2})(\tilde{u}_{f_1} \gamma_0 u_{i1})$$

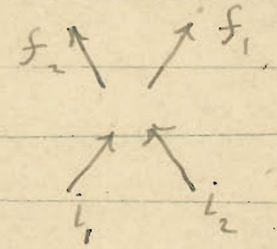
Note $\frac{4\pi}{Q^2-q^2} = \int e^{i(Qt - q\cdot r)} \left[\frac{1}{2r} \{ \delta(r-t) + \delta(r+t) \} \right] d^4 r$

which is a purely classical delay effect.

The two processes

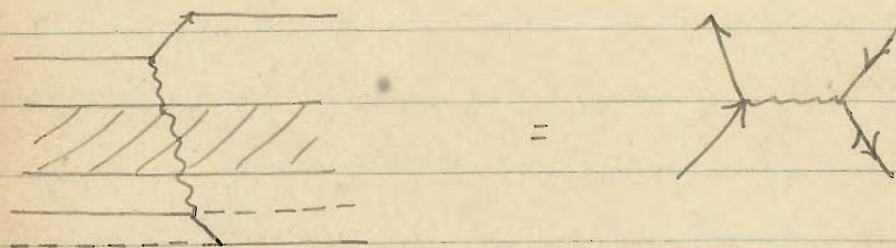


and



can not be distinguished. Their matrix elements must be added with a minus sign introduced to take care of the electron exchange involved.

A similar formula holds for electron positron scattering. In this case we have the two processes

I 

II