Mathematical Methods
Feynman

Lecture I
Oct 14, 1946

References:

(1) Methods of Advanced Calculus
P. Franklin

(2) Mathematics of Physics & Chemistry
Margenau & Murphy

(3) Table of Integrals
Pierce & Dwight

I Elementary Mathematics

A Equations

The simplest way to solve any equation - except linear or quadratic equations - is by <u>trial and error</u>

(1) Interpolation

$1/1+x^2 = 2x$

x	LHS	RHS	DIFF	
0	1	0	1	} 2.5
1	.5	2	-1.5	}
.4	.862	.8	.062	
.5	....	....	....	

(2) Iteration

Solve the equation for X

$X = \frac{1}{2}(1/1+x^2)$

trial X	result X
0 →	.5
.5 ↙	.4
.4 ↙	.431
.... ↙	....

If the equation is solved in the form $X = \sqrt{\frac{1}{2x} - 1}$, however, the results diverge. If the results do diverge solving the equation in the reverse manner will make them converge.

(3) Newtons Method

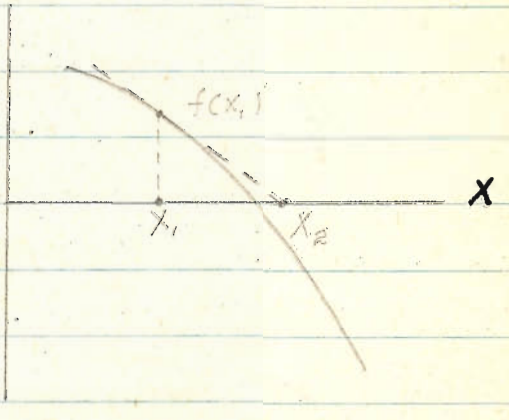
Solve the equation in the form $f(x) = 0$. Let x_1 be the first trial, then the second trial is given by the relation

$$x_2 = x_1 - f(x_1) / f'(x_1)$$

This procedure is equivalent to extrapolating back along the tangent. If the curve is full of wiggles things may diverge.

$f(x) = 1/1+x^2 - 2x = 0$

$f'(x) = -2x/(1+x^2)^2 - 2$



x	f(x)	f'(x)
0	1	-2
.5	.2	....
....	....	....

Problem: Find first positive root of $e^{-x} = \cos x$ by all three methods to 2 decimals. Estimate higher roots.

B Power Series

Problems: Prove

$$1 + X + X^2 + X^3 + \ldots = 1/1-X$$

Sum

$$1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \ldots$$

A Summation by Integration and Differentiation

$$1(X) - \tfrac{1}{2}(X^2) + \tfrac{1}{3}(X^3) - \tfrac{1}{4}(X^4) + \ldots = S(X)$$

$$1 - X + X^2 - X^3 + X^4 + \ldots = S'(X)$$

$$S'(X) = 1/1+X$$

$$S(X) = \ln(1+X) + C$$

evaluate constant for $X = 0$

$$S(0) = \ln 1 + C$$

$$0 = 0 + C$$

$$S(1) = \ln 2 = .69315$$

B Convergence: Ratio Test

If the

$$\lim_{n \to \infty} X_{n+1} / X_n < 1$$

the series converges.

Math Methods

Lecture II Oct 16, 1946

I Series:

A Abel Summability.

Consider the series $1-2+3-\ldots$

Convert it into a power series in x as follows:

$$S(x) = 1 - 2x + 3x^2 - \ldots$$

Sum this power series:

$$\int S(x)\,dx = x - x^2 + x^3 - \ldots = x/1+x$$

$$S(x) = \frac{d}{dx}(x/1+x) = 1/(1+x)^2$$

The Abel sum of the original series is defined as the $\lim_{x \to -1} S(x)$

$$\text{Abel sum} = \lim_{x \to -1} [1/(1+x)^2] = 1/4$$

In general, oscillating series can be summed, but the order of the terms must not be changed.

B Summation of Series by Integration and Differentiation.

$$S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

$$S(x) = x + x^2/4 + x^3/9 + \dots$$

$$S'(x) = 1 + x/2 + x^2/3 + \dots$$

$$= -\ln(1-x)/x$$

$$S(x) = -\int [\ln(1-x)/x]\, dx$$

One should know and recognize the series for the following expressions:

(1) $1/1-x = 1 + x + x^2 + x^3 + \dots$

(2) $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

→ (3) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

(4) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(5) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

(6) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

(7) $1/(1+x)^2 = 1 - 2x + 3x^2 - \ldots$

(8) $1/(1+x)^{\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{15}{48}x^3$

An useful relation $e^{i\theta} = \cos\theta + i\sin\theta$

Problem II Sum:

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots = 1$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots = 1$$

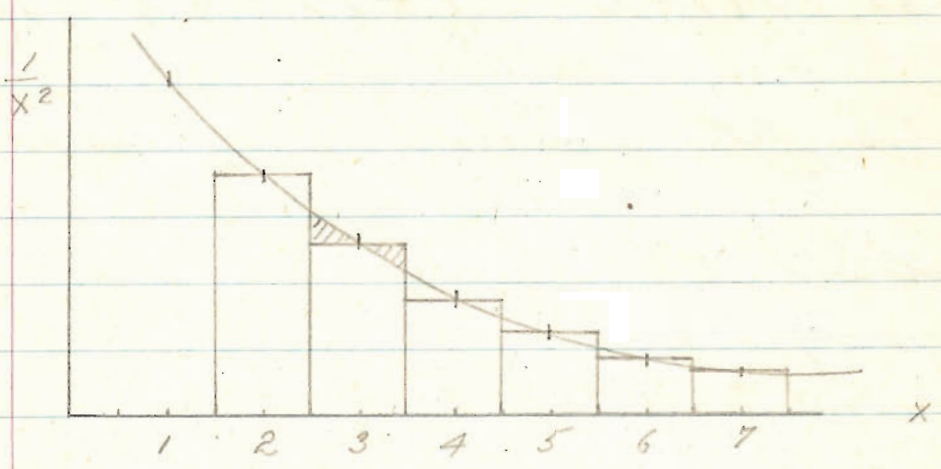
C Summation of Series by Adding Terms.

Consider the series $1 + \frac{1}{4} + \frac{1}{9} + \ldots$
We can sum it as follows:

1.000
.250
.111
.0625
.....

By taking enough terms we can obtain any desired accuracy. If the series is slow to converge it is sometimes

possible to replace the higher terms by an integral:



Examination of the above diagram shows that

$$\sum \frac{1}{3^2} + \frac{1}{4^2} + \ldots \approx \int_{2\frac{1}{2}}^{\infty} \frac{1}{x^2}\, dx$$

$$\approx \frac{1}{2.5} = .4$$

The sum of our series is then

1.000
.250
.400
1.650

Since the error will be principally in the first terms some idea of its magnitude

may be gotten by looking at the next best approximation:

$$\sum \frac{1}{4^2} + \frac{1}{5^2} + \ldots \approx \int_{3\frac{1}{2}}^{\infty} \frac{1}{x^2}\, dx$$

$$\approx .286$$

The new sum is:

1.000
.250
.111
.286
1.647

we see that we have made an error of only .003. We can guess from this that the third figure is probably about 6.

Magnitude of Quantities

The following quantities are arranged in the order of their rate of rise for large x

$$1 < \ln x < x^{\epsilon} < x < x^n < e^x$$

To ascertain their behavior for

small y make the substitution $y = \frac{1}{x}$

The following approximations hold for small x.

$$\sin x \sim x$$
$$\cos x \sim 1 - \frac{x^2}{2}$$
$$\tan^{-1} x \sim x \sim \tan x$$
$$\sinh \sim x$$
$$e^x \sim 1 + x$$
$$(1+x)^n \sim 1 + nx$$
$$(1+x)^{-\frac{1}{2}} \sim 1 - \frac{1}{2}x$$

E De L'Hospital's Rule

If as $x \to a$, $f(x)$ and $g(x) \to 0$

$$\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x)$$

Example: $\tan x / [(1+x)^3 - 1]$ $\quad x = 0$

$$= \sec^2 x / 3(1+x)^2 = \frac{1}{3}$$

Math Methods

Lecture III Oct 18, 1946

Problem III Find $\lim x \to 0$

(a) $\sin x / (e^{3x} - 1)$

(b) $\ln(\frac{1}{2} + 1/2\sqrt{1+x^2}) / (e^{x^2} - \cos x^2)$

Give an approximate expression for

$(\sin kr / r^2) - (k \cos kr / r)$

for small r.

I Integration

A Methods

(1) Substitution

(2) By Parts

(3) Differentiation of a Parameter

(4) Series Expansion

(5) Contour Integration

(6) Numerical Methods

(7) Special Tricks

Methods 3 to 7 are applicable particularly to definite integrals.

A Complex Variable in Substitution

Find:

$$\int_0^\infty e^{-ax} \cos bx \, dx$$

$$\cos bx = (e^{ibx} + e^{-ibx})/2$$

$$= \frac{1}{2}\int_0^\infty e^{-(a-ib)x} dx + \frac{1}{2}\int \ldots$$

$$= \frac{1}{2}\frac{1}{a-ib} + \frac{1}{2}\frac{1}{a+ib}$$

$$= a/a^2+b^2$$

B Differentiation of a Parameter

Example: $\int_0^\infty x e^{-ax} \cos bx$

$$S(a) = \int_0^\infty e^{-ax} \cos bx = a/a^2+b^2$$

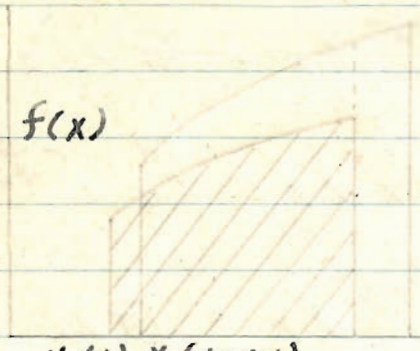
Differentiate with respect to a

$$S'(a) = -\int_0^\infty x e^{-ax} \cos bx = \frac{(b^2-a^2)}{(a^2+b^2)^2}$$

The general rule for differentiation with respect to a parameter is:

$$\frac{d}{da}\left[\int_{x_1(a)}^{x_2(a)} f(x,a)\,dx\right] = \int_{x_1}^{x_2} \frac{\partial}{\partial a} f(x,a)\,dx$$

$$+\left[\frac{d}{da}x_2(a)\right] f(x_2,a) - \left[\frac{d}{da}x_1(a)\right] f(x_1,a)$$



$$\Delta \int_{X_1}^{X_2} f(x,\alpha)\,dx = \int_{X_1}^{X_2} \Delta f(x,\alpha)\,dx$$

$$+ \Delta X_2(\alpha)\, f(X_2,\alpha) - \Delta X_1(\alpha)\, f(X_1,\alpha)$$

c Multiplication by a Factor

Example: $\int_0^\infty (\sin x / x)\,dx$

$$\int_0^\infty e^{-\alpha x} \frac{\sin x}{x}\,dx = S(\alpha)$$

$$S'(\alpha) = -\int_0^\infty e^{-\alpha x} \sin x\,dx = -1/(1+\alpha^2)$$

$$S(\alpha) = -\tan^{-1}\alpha + C$$

$$S(\infty) = 0 = -\frac{\pi}{2} + C$$

$$S(\alpha) = \frac{\pi}{2} - \tan^{-1}\alpha$$

$$S(0) = \frac{\pi}{2}$$

Problem IV

(a) Given $\int_0^\infty e^{-a^2x^2}\,dx = \sqrt{\pi}/2a$

Prove: $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}}$

(b) Given $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$

Find: $\int_0^\infty x^4 e^{-x^2}\,dx$

(c) $\int_0^\infty dy\,(e^{-ay} - e^{-by})/y$

(d) $\int_{-\infty}^\infty e^{-Bx^2 - \frac{\alpha}{x^2}}\,dx$

(e) $\int_0^\infty dx\,\sin^2 x / x^2$

Math Methods

Lecture IV Oct 21, 1946

I Differentiation under Integral Sign

A. $$\int_0^\infty e^{-ax}(1+x^2)^{-1}\,dx = S(a)$$

$$\int_0^\infty x^2 e^{-ax}(1+x^2)^{-1}\,dx = S''(a)$$

$$S(a) + S''(a) = \int_0^\infty \left(\frac{1}{1+x^2} + \frac{x^2}{1+x^2}\right) e^{-ax}\,dx$$

$$\frac{d^2S}{da^2} + S = \frac{1}{a}$$

This differential equation may be solved for $S(a)$

B. Partial Differentiation

$$S(\alpha,\beta) = \int_0^\infty e^{-ax}\sin\beta x\,dx$$

$$\frac{\partial^2 S}{\partial \beta^2} = -\int_0^\infty x^2 e^{-ax}\sin\beta x\,dx$$

$$\frac{\partial^2 S}{\partial \alpha^2} = \int_0^\infty x^2 e^{-ax}\sin\beta x\,dx$$

$$\frac{\partial^2 S}{\partial \beta^2} = -\frac{\partial^2 S}{\partial \alpha^2}$$

The form of S may be partly determined by making the substitution $y = \beta x$

$$S(\alpha,\beta) = \beta^{-1}\int_0^\infty e^{-\frac{\alpha}{\beta}y}\sin y\,dy = \beta^{-1}F\left(\frac{\alpha}{\beta}\right)$$

$$\frac{\partial}{\partial\beta}\left[\beta^{-1}F\left(\frac{\alpha}{\beta}\right)\right] = \beta^{-1}F'\left(-\frac{\alpha}{\beta^2}\right) - F\beta^{-2}$$

$$\frac{\partial^2}{\partial\beta^2} \quad = -\frac{\alpha}{\beta^3}F''\left(-\frac{\alpha}{\beta^2}\right) + F'\frac{3\alpha}{\beta^4} \quad F\frac{2}{\beta^3} - \beta^{-2}F'\left(-\frac{\alpha}{\beta^2}\right)$$

$$\frac{\partial^2 S}{\partial\beta^2} = \frac{2}{\beta^3}F + \frac{4\alpha}{\beta^4}F' + \frac{\alpha^2}{\beta^5}F''$$

$$\frac{\partial^2 S}{\partial\alpha^2} = \frac{1}{\beta^3}F''$$

Multiply by β^3 and let $z = \frac{\alpha}{\beta}$

$$2F + 4zF' + z^2F'' = -F''$$

$$\frac{d}{dx}(z^2F') + 2\frac{d}{dx}(zF) = -F'$$

$$z^2F' + 2zF = -F' + C_1$$

$$\frac{d}{dx}(z^2F) = -F' + C_1$$

$$z^2F = -F + C_1 z + C_0$$

$$F = (C_0 + C_1 z)/(1 + z^2)$$

$$S(\alpha,\beta) = \frac{1}{\beta}F = (C_0\beta + C_1\alpha)/(\alpha^2 + \beta^2)$$

To evaluate C_0 & C_1 we observe:

$$S(\alpha, 0) = 0 = \frac{C_1 \alpha}{\alpha^2} \quad \therefore \; C_1 = 0$$

For small β, $\sin \beta x \sim \beta x$. This is a good approximation since $e^{-\alpha x}$ kills the integrand for large x.

$$\int_0^\infty e^{-\alpha x} \beta x \, dx \equiv \frac{\beta}{\alpha^2}$$

$$\frac{C_0 \beta}{\alpha^2 + \beta^2} \cong \frac{C_0 \beta}{\alpha^2} \quad \therefore \; C_0 = 1$$

$$S(\alpha, \beta) = \frac{\beta}{\alpha^2 + \beta^2}$$

II The Dirac Delta Function

$\delta(x) = 0$, $x \neq 0$, and is in some way infinite for $x = 0$

$$\int_{-\infty}^{\infty} \delta(x) \, dx \equiv \int_{-\epsilon}^{\epsilon} \delta(x) \, dx \equiv 1$$

$$\int_{-\infty}^{\infty} f(x) \, \delta(x) \, dx = f(0)$$

Proof: $\int_{-\infty}^{-\epsilon} f(x) \, \delta(x) \, dx + \int_{-\epsilon}^{\epsilon} f(x) \, \delta(x) \, dx$

$$+ \int_{\epsilon}^{\infty} f(x) \, \delta(x) \, dx$$

If $f(x)$ varies relatively slowly in the range $-\epsilon$ to ϵ we can replace it by an average of $f(x)$ over that interval.

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = \int_{-\epsilon}^{\epsilon} \overline{f(x)}\,\delta(x)\,dx$$

$$= \overline{f(x)}\Big|_{x=0} \cdot \int_{-\epsilon}^{\epsilon} \delta(x)\,dx$$

$$\epsilon \to 0 \qquad = f(0)$$

The delta function is useful since it may be operated on as tho it were a real function. The only justification we give for this is that it gives the correct answer.

$$f'(t) = -\int_{-\infty}^{\infty} f(x)\,\delta'(x-t)\,dx$$

$$= f(x)\,\delta(x-t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x-t)\,dx$$

$$= f'(t)$$

The following useful relations

may be easily proved:

(a) $\delta(x) = \delta(-x)$

(b) $\delta'(x) = -\delta'(-x)$

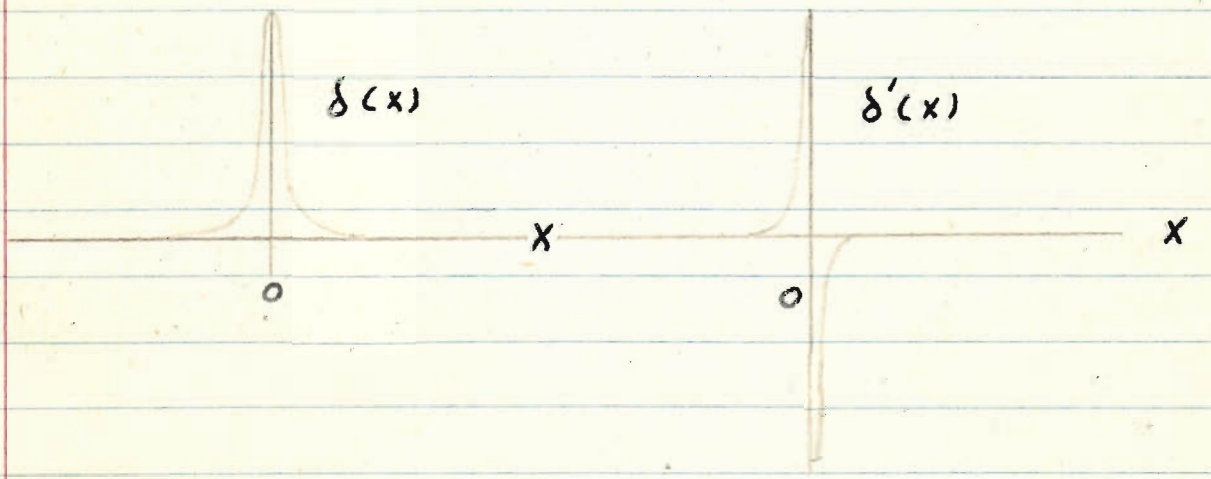
(c) $x\,\delta(x) = 0$

(d) $x\,\delta'(x) = -\delta(x)$

(e) $\delta(ax) = \frac{1}{|a|}\,\delta(x)$

(f) $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$

The following figures are approximate representations of $\delta(x)$ & $\delta'(x)$



Reference: Q.M. Dirac pp 72 to 77

Problem V (with integrals)

(a) Prove: $\delta(ax) = \frac{1}{a}\delta(x)$
for $a > 0$

(b) Find: $x\,\delta''(x) = ?$

Math Methods

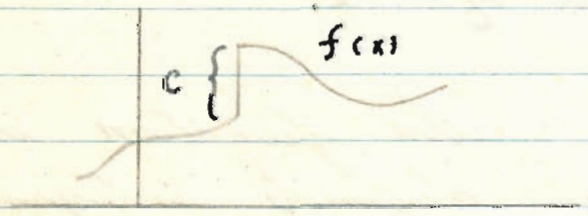
Lecture V Oct 21, 1946

I Dirac Delta Function

A. Step Function:

$$\int_{-\infty}^{x} \delta(x)\,dx = 0 \quad x < 0$$
$$= 1 \quad x > 0$$

Consider the curve in figure 1. Its derivative may be expressed in terms of the delta function as follows:



$$f'(x) = f'(x)_{x \neq a} + c\,\delta(x-a)$$

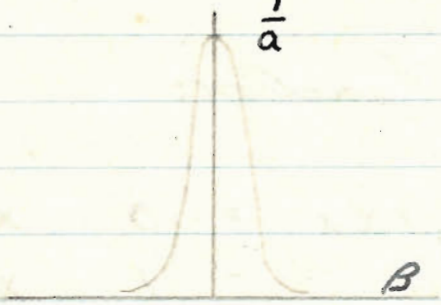
fig 1

Consider the integral: $\int_{-\infty}^{\infty} \cos \beta x\,dx$. We shall show that it is equal to $2\pi\,\delta(\beta)$.

$$\rightarrow \int_{0}^{\infty} \cos \beta x\,dx = \pi\,\delta(\beta)$$

$$\lim_{a \to 0} \int_0^\infty e^{-ax} \cos \beta x \, dx = \lim_{a \to 0} \frac{a}{a^2 + \beta^2}$$

For small a this integral looks approximately as in figure 2.



The area under the curve is:

$$A = \int_{-\infty}^{\infty} \left(\frac{a}{a^2 + B^2} \right) dB$$

$$\beta = az$$

$$= \int_{-\infty}^{\infty} \frac{a^2}{a^2 + a^2 z^2} \, dz$$

$$= \int_{-\infty}^{\infty} \frac{dz}{1 + z^2} = \tan^{-1} z \Big|_{-\infty}^{\infty} = \pi$$

Note that the area is independent of the value of a.

Use of the delta function in evaluating a definite integral:

$$\int_0^{\frac{\pi}{2}} \cos(m \tan \theta) \, d\theta$$

Substitute $\tan \theta = x$

$$\int_0^\infty \cos mx \, (1 + x^2)^{-1} \, dx = S(m)$$

$$-\int_0^\infty x^2 \cos mx \, (1 + x^2)^{-1} \, dx = S''(m)$$

$$S(m) - S''(m) = \int_0^\infty \cos mx \, dx = \pi \delta(m)$$

The general solution of such a differential equation is:

$$S(m) = \frac{e}{2}\int_{-\infty}^{m} e^{m-t} f(t)\,dt - \frac{e}{2}\int_{-\infty}^{m} e^{t-m} f(t)\,dt + Ae^{m} + Be^{-m}$$

$$f(t) = \pi\,\delta(t)$$

Solutions:

$m < 0 \qquad S = Ae^{m} + Be^{-m}$

$m > 0 \qquad S = Ae^{m} + Be^{-m} + \frac{\pi}{2}e^{m} + \frac{\pi}{2}e^{-m}$

$$S(0) = \frac{\pi}{2} = A + B$$

$$S(m) = S(-m) \quad \therefore \quad A - B = \frac{\pi}{2}$$

$$A = \frac{\pi}{2}, \quad B = 0$$

$m < 0 \qquad S = \frac{\pi}{2}e^{m}$

$m > 0 \qquad S = \frac{\pi}{2}e^{-m}$

II Investigation of the Existance of Integrals.

Consider: $\int_0^{\infty} \frac{e^{-ax}}{x}\,dx$

Near 0

$$\sim \int_0 \frac{dx}{x} = -\ln 0 + \text{number}$$

The integral is infinite.

Consider: $\int_0^\infty \frac{1-e^{-ax}}{x}\,dx$

$X \sim 0 \qquad \int_0 a\,dx = 0 + \ldots$

$X \to \infty \qquad \int^\infty \frac{dx}{x} = \ln \infty + \ldots$

Again the integral is infinite

The following table of functions may prove useful in determining whether or not a given integral exists.

	$X \to \infty$	$X \to 0$
$\frac{dx}{x^2}$	O. K.	N. G.
$\frac{dx}{x}$	N. G	N. G.
$\frac{dx}{x^{1+\epsilon}}$	O. K.	N. G
$\frac{dx}{x^{1-\epsilon}}$	N. G.	O. K.
dx	N. G.	O. K

Problem: The $\int \frac{1-e^{-ax^2}}{x}\,dx$ exists. Find its value.

Math Methods

Lecture VII Oct 25, 1996

Problems: Integrate:

(1) $\int_0^\infty \frac{\cos mx}{(1+x^2)^2}$

Prove:

(2) $\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\pi/1}$

Integrate

(3) $\int_0^\infty \frac{x\,dx}{e^{mx} - e^{-mx}}$

(4) $\int_0^\infty \frac{\sin x\,dx}{\sinh ax} = S(a)$

(a) Find $S(1)$ to 3 significant figures

(b) Find approximate expressions for $S(a)$ for large and small a.

I Special Method of Evaluating a Definite Integral. (This method is applicable to problem 2)

$$A = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$A^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$A^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy$$

Substitute $r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$

$$A^2 = 4\int_0^{\infty}\int_0^{\pi/2} r e^{-r^2} dr\,d\theta$$

$$= \pi \int_0^{\infty} 2r e^{-r^2} dr = -\pi e^{-r^2}\Big|_0^{\infty}$$

$$A = \sqrt{\pi}$$

II Series Solution of Differential Equations.

Consider:

$$\int_0^{\infty} \frac{dx}{e^{nx}+e^{-nx}} = \frac{\pi}{4n}$$

$$\int_0^{\infty} \frac{e^{-nx}dx}{(1+e^{2n})} = \int_0^{\infty} e^{-nx}dx\,(1 - e^{-2nx} + e^{-4nx} - \dots)$$

$$= \int_0^{\infty} dx\,(e^{-nx} - e^{-3nx} + e^{5nx} - \dots)$$

$$= \int_0^{\infty} e^{-nx}dx - \int_0^{\infty} e^{-3nx}dx + \int_0^{\infty} e^{5nx}dx - \dots$$

$$= \left(\frac{1}{n} - \frac{1}{3n} + \frac{1}{5n} - \ldots\right) = \left(\frac{1}{n}\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right)\right.$$

$$\therefore \int_0^\infty \frac{dx}{e^{nx} + e^{-nx}} = \frac{1}{n}\left(\frac{\pi}{4}\right)$$

A combination of the methods of integration discussed this far is sufficient to to solve any integral which exists. There are still other methods which may be used and are in many cases easier. We shall discuss some of them later.

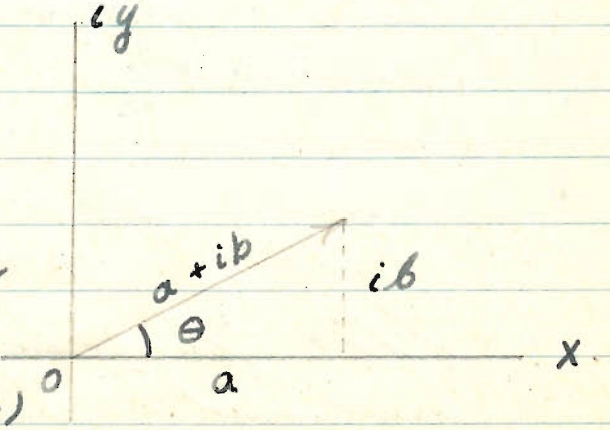
III Complex Variables.

General Review

$A = a + bi$

$B = c + di$

Rule of Addition



(1) $A + B = (a + c) + i(d + b)$

Rule of Multiplication

$$AB = (ac - bd) + i(ad + bc)$$

A complex number may also be represented as follows

$$A = (a + bi) = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}} + \frac{ib}{\sqrt{a^2+b^2}}\right]$$

$$A = r(\cos\theta + i\sin\theta)$$

(2) $$AB = rr'(\cos(\theta+\theta') + i\sin(\theta+\theta'))$$

$$A^n = r^n(\cos n\theta + i\sin n\theta)$$

Suppose we let $r + r'$ in equation (2) equal unity. We have an equation of the type:

$$f(\theta)f(\theta') = f(\theta+\theta')$$

This function equation is solved by

$$f(\theta) = R^\theta$$

$$R^{\theta} R^{\theta'} = R^{\theta+\theta'}$$

$$R^{\theta} = \cos\theta + i \sin\theta$$

We have in addition:

$$R^{\theta} \times \bar{R}^{\theta} = 1$$

$$R \times \bar{R} = 1$$

$$R = 1/\bar{R}$$

This condition is satisfied if $R = e^{i}$.

(3) $$e^{i\theta} = \cos\theta + i \sin\theta$$

This equation may be used as the defining equation for e. Note that $e^{i\theta}$ has period 2π

From equation (3), we learn:

$$e^{i\pi a'} + 1 = 0 \qquad a' = 1, 3, 5, \ldots$$

$$e^{i2\pi a} - 1 = 0 \qquad a = 0, 1, 2, \ldots$$

n^{th} Roots of Unity

$$\sqrt[n]{1} = \left(e^{i2\pi a}\right)^{\frac{1}{n}}$$

There will be n distinct

roots corresponding to values of "a" from 0 to $n-1$.

Logarithm of a Complex Number.

$$\ln(a+bi) = \ln(re^{i\theta})$$
$$= \ln r + i\theta$$
$$= \ln\sqrt{a^2+b^2} + i\tan^{-1}\frac{b}{a}$$
$$= \ln\sqrt{a^2+b^2} + i\tan^{-1}\frac{b}{a} + 2\pi k \quad k = 1, 2, 3,$$

The logarithm of a complex number is a many valued function.

Problem Prove: $\ln(-1) = i\pi$

Math Methods

Lecture VII Oct 28, 1946

I More About the Delta Function

Let us consider the integral

$$\int_{-\infty}^{\infty} \frac{\sin \beta x}{x}\, dx = S(\beta)$$

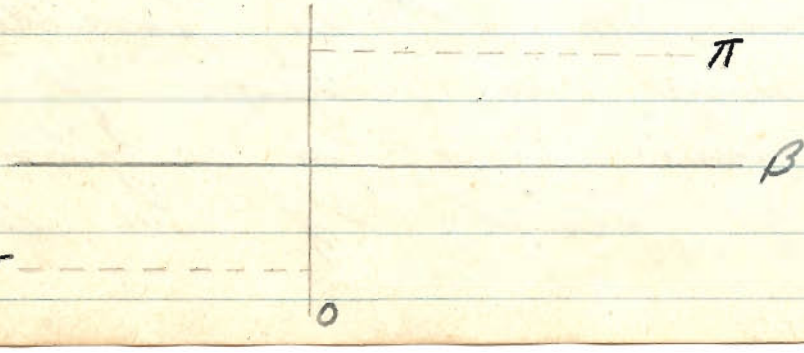
$$S'(\beta) = \int_{-\infty}^{\infty} \frac{x \cos \beta x}{x} = 2\pi\, \delta(\beta)$$

$$S(\beta) = 2\pi + C \qquad \beta > 0$$
$$S(\beta) = C \qquad \beta < 0$$

We note that $S(\beta) = -S(-\beta)$

$$2\pi + C = -C$$
$$C = -\pi$$

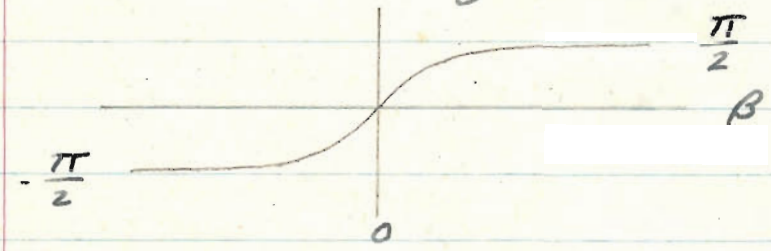
$$\beta > 0 \qquad S(\beta) = \pi$$
$$\beta < 0 \qquad S(\beta) = -\pi$$



Consider now the integral

$$\int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx = \frac{\pi}{2} - \tan^{-1}\left(\frac{\alpha}{\beta}\right)$$

A graph of this integral looks as follows:



In the $\lim \alpha \to 0$, this integral reduces to the one considered previous to it (there is a factor 2 to be considered)

II Conformal Mapping

The equation $W = f(Z)$ establishes a relation between the points of the W and Z planes. By using such a relation it is possible to map a region of the Z plane into a region of the W plane. If in this

process neighborhoods go into neighborhoods the mapping is said to be continuous. Points at which this is not true are called singularities.

Let us examine a few such maps.

(a) $W = a Z$ Multiplication by a real number equals a change of scale.

(b) $W = e^{i\theta} Z$ Multiplication by $e^{i\theta}$ equals a positive rotation thru the angle θ.

(c) Multiplication by a complex number $A = r e^{i\theta}$ equals a scale change by r and a rotation thru the angle θ.

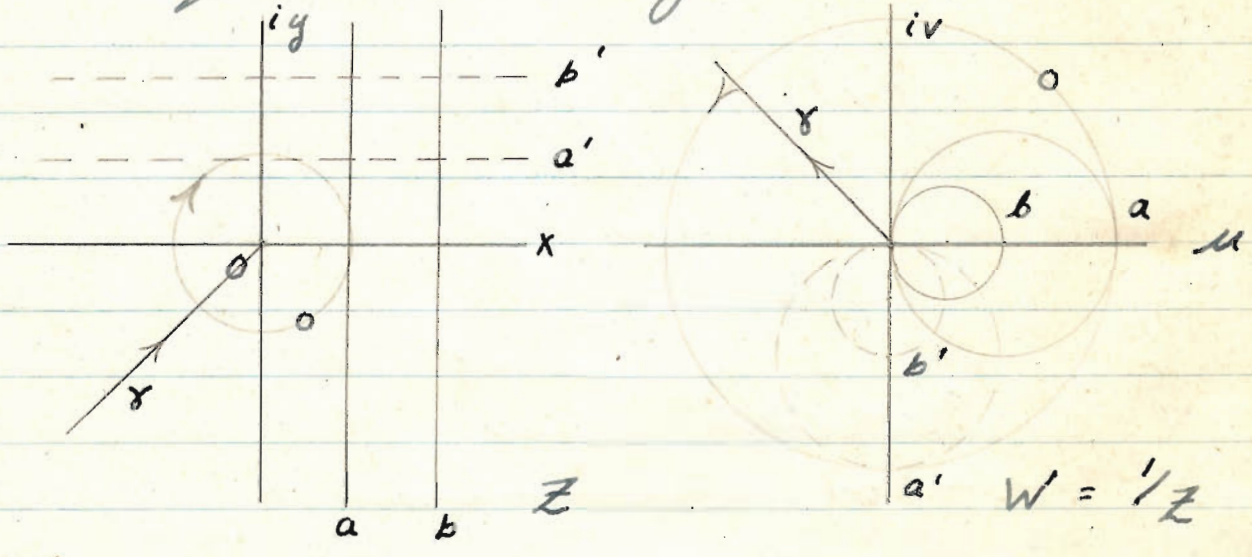
(d) Addition of a complex number B equals a translation by displacement B

(e) $W = \bar{Z}$ equals a reflection on the real axis.

Consider the function:

$$W = 1/Z$$

This corresponds to an inversion and reflection on the axis of reals. A pure inversion is represented by $W = 1/Z$



The function $1/Z$ is said to have a "simple pole" at the origin.

Math Methods

Lecture VIII Oct 30, 1946

I Complex Variables

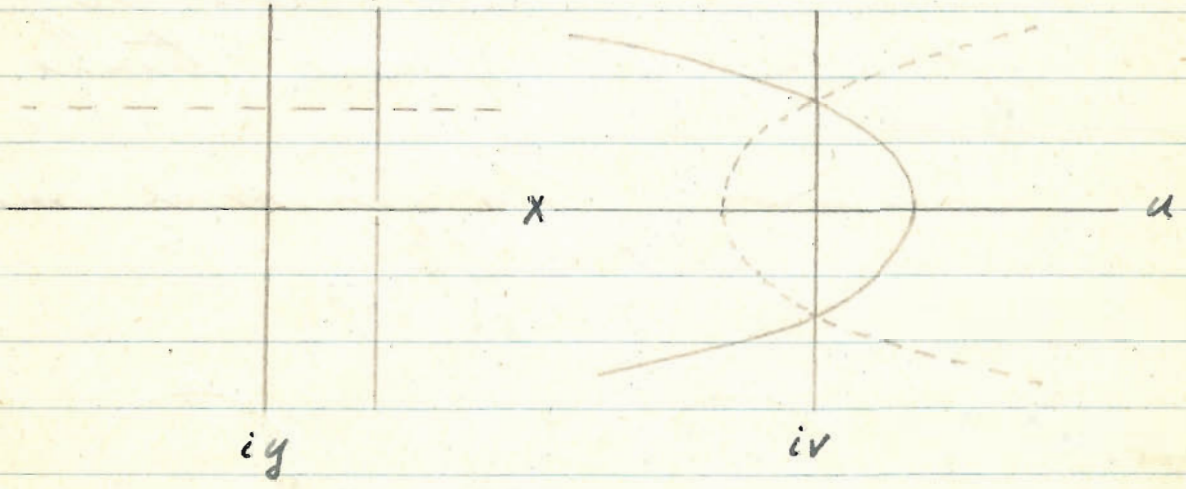
A Conformal Mapping.

Consider the function

$$W = Z^2$$

$$Z = re^{i\theta}$$

$$Z^2 = r^2 e^{2i\theta}$$



Analytic Treatment

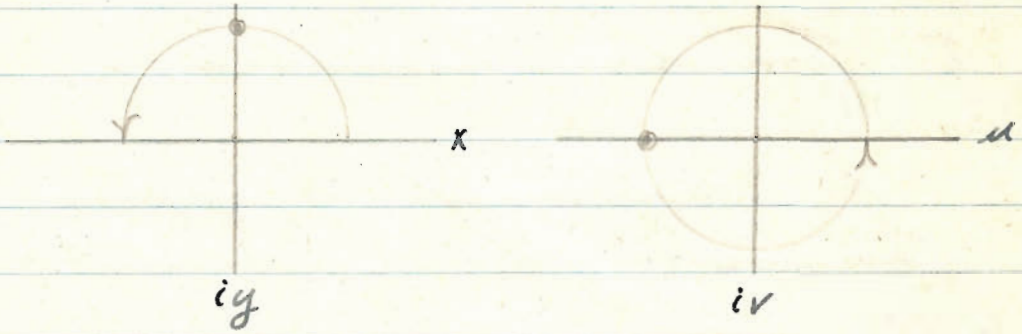
$$Z = 1 + iy$$

$$Z^2 = 1 - y^2 + 2iy$$

$$u = 1 - y^2 \qquad v = 2y$$

$$u = 1 - \frac{v^2}{4}$$ Parabola

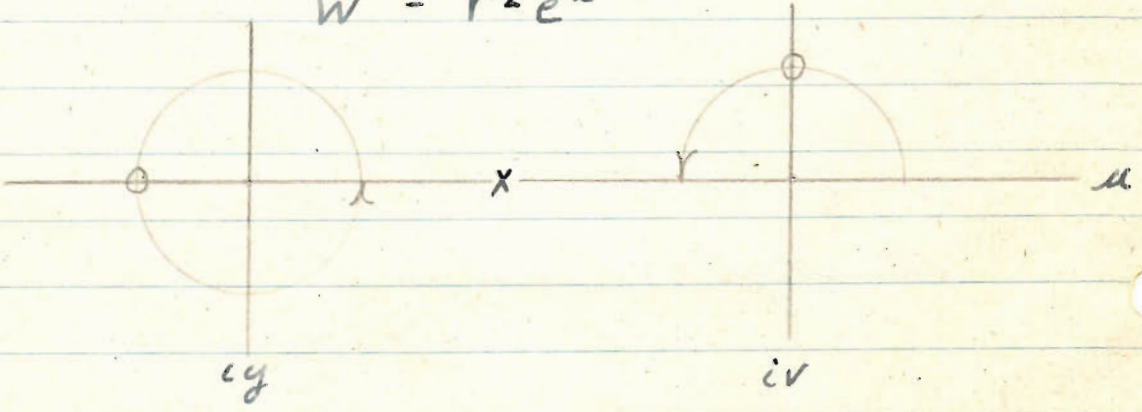
Let us what happens as we circle the origin.



We observe that in circling the origin of the Z plane once, we circle the origin of the W plane twice.

Consider the inverse map:

$$W = Z^{\frac{1}{2}}$$
$$W = r^{\frac{1}{2}} e^{\frac{1}{2} i\theta}$$



In this case, if we circle the origin of the Z plane once, we only get half

way around the origin of the W plane. This is a consequence of the fact that we have neglected the negative root of $Z^{\frac{1}{2}}$. The origin is in this case termed a branch point.

Branch Point

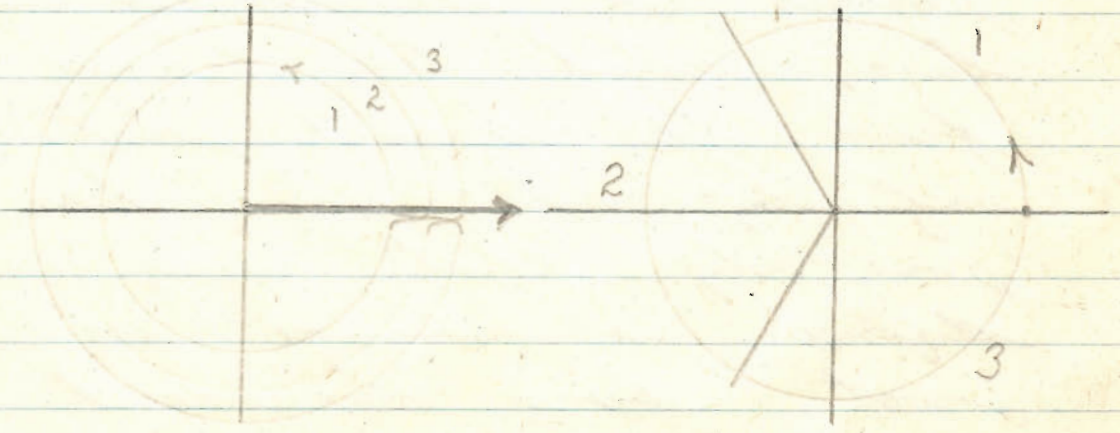
A branch point in the Z plane is a point which if it is circled once defines a curve in the W plane which does not close on itself. In order to keep track of the roots of functions having branch points we draw an imaginary line, called the branch line, from the branch point to infinity. This line has the property that every crossing defines a change in the root being considered. The way in which a branch line

operatus is best pictured geometrically. We imagin the Z plane made of as many layers of paper as the function has roots. These layers are joined in such a way that in profile they look like this:

$W = Z^{\frac{1}{2}}$ $W = Z^{\frac{1}{3}}$



The intersections are the branch lines. The rule is that when ever one is on a given layer one uses the root associated with that layer.



Math Methods

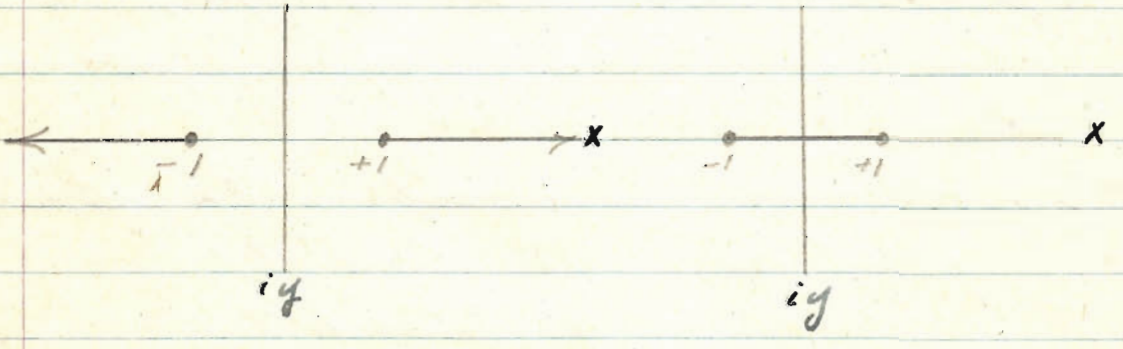
Lecture IX November 1, 1946

I Complex Variables

A. Branch Lines:

Consider the function $W = \frac{1}{\sqrt{Z^2 - 1}}$

This function has branch points at +1 and -1. The branch lines may be represented in either of two ways.

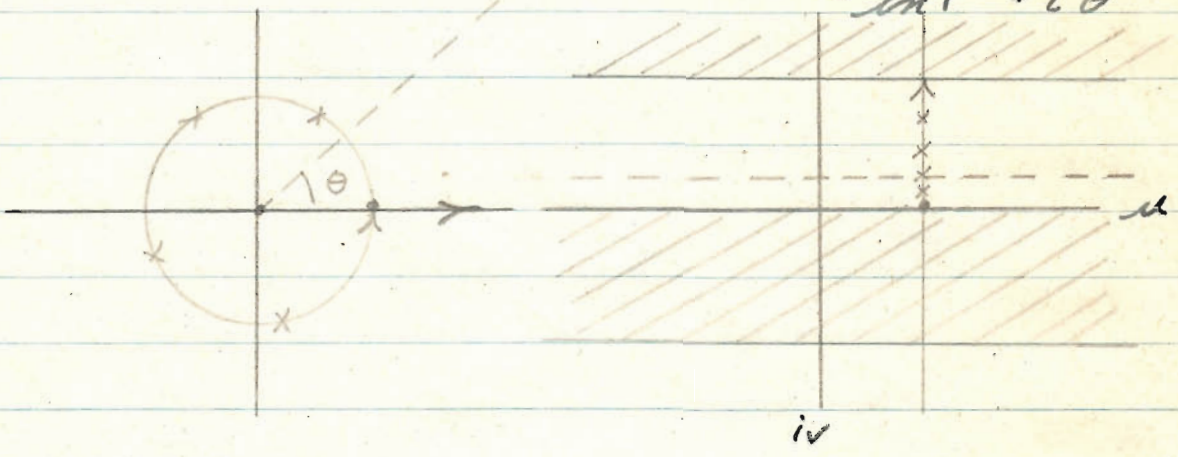


If we imagin that the branch lines in figure 1 join at infinity it is easy to see that the two representations are equivalent, since the branch line may

"blow in the wind" as it were.

B The Logarithm $W = \ln Z$

$= \ln r + i\theta$



The logarithm has a brand point at the origin, and has infinitely many branches.

→ Problems: Describe the following functions: 1) $\tan^{-1} Z$

2) $\tanh Z$

3) $1/\sqrt{Z^2-1}$

4) $\sin Z$

Where are the singularities and what kind are they.

Math Methods

Lecture X Nov 4, 1946

I Complex Variables

A. The Derivation

$$\frac{df(z)}{dz} = \lim_{\Delta z \to 0} f(z+\Delta z) - f(z) \,/\, \Delta z$$

In general, the value of this derivation will depend on the way in which Δz approach zero. We illustrate this by an example:

$$z = x + iy$$
$$f(z) = x - iy$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \{[(x+\Delta x) - i(y+\Delta y)] - (x - iy)\} / (\Delta x + i\Delta y)$$

$$= \lim_{\Delta z \to 0} (\Delta x - i\Delta y) / (\Delta x + i\Delta y)$$

If we let Δx then Δy go to zero, $df/dz = -1$. If we let Δy then Δx go to zero, $df/dz = 1$

We see that the value of the derivative depends on the path by which $\Delta z \to 0$.

Problem: Investigate value of df/dz for $f(z) = x - iy$ for paths at constant θ.

B. Analytic Functions of a Complex Variable.

An analytic function of a complex variable is usually defined as one whose derivative is independent of the path by which $\Delta z \to 0$. Let us examine the conditions which must hold if this is to be the case.

$$f(x+iy) = u + iv$$

$y \to 0$
$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$x \to 0$
$$\frac{df}{dz} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x}\right]$$

If $\frac{df}{dz}$ is to be independent of

the path by which $\Delta z \to 0$, these two expressions for the derivative must be equal.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Hence: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

The above equations imply preservation of angles. The above equations imply the following equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

These are simply Laplace Equations in two dimensions. We can see that analytic functions of a complex variable may be used in the solution of physical problems where the two dimensional Laplace Equation holds.

C. Integral of a Complex Function.

$$\int_{z_1}^{z_2} f(z)\,dz = \lim_{\to 0} \sum f(z)\,\Delta z$$

This integral will, in general, depend on the path of integration. Let examine the conditions under which the integral will be independent of the path of integration.

$$\int f(z)\,dz = \int (u\,dx - v\,dy) + i \int (v\,dx + u\,dy)$$

The integrals on the right will be independent of the path of integration if the integrands are perfect differentials:

$$u\,dx - v\,dy = d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy$$

$$u = \frac{\partial \phi}{\partial x} \; ; \; v = -\frac{\partial \phi}{\partial y}$$

Hence: $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$; $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Math Methods

Lecture XI Nov 6, 1946

I Discussion of Mapping Problem

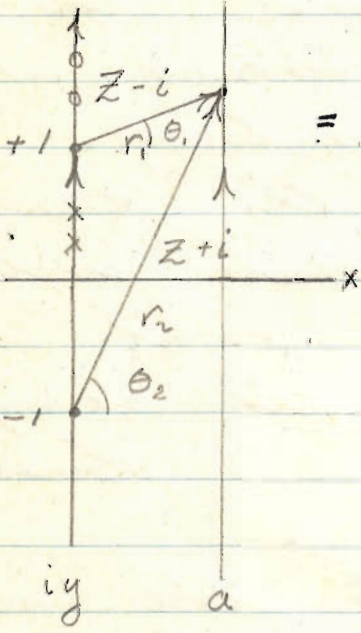
$$W = \tan^{-1} Z$$

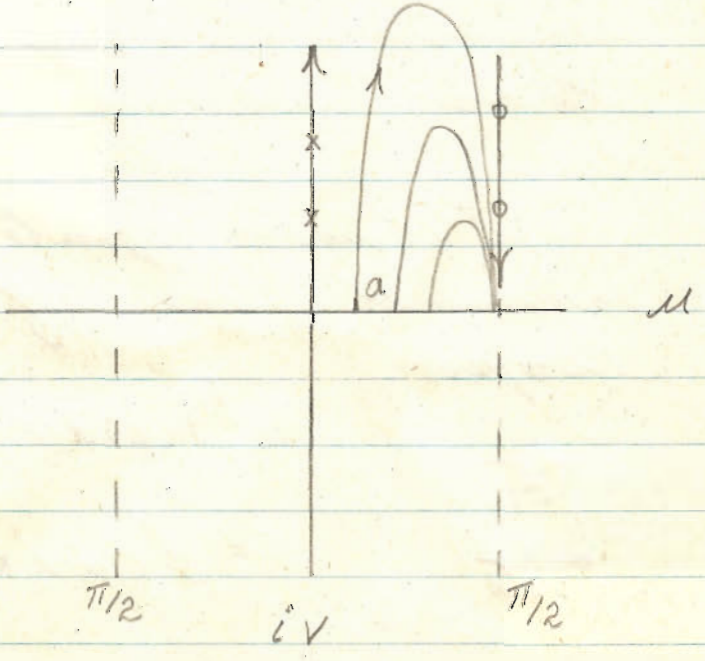
$$= i \tanh^{-1} iZ$$

$$= \frac{i}{2} \ln \left(\frac{Z+i}{Z-i} \right) + \frac{\pi}{2}$$

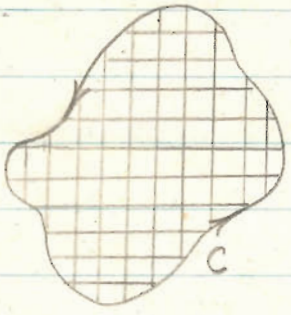
$$= \frac{i}{2} \ln \frac{r_2 e^{i\theta_2}}{r_1 e^{i\theta_1}} + \frac{\pi}{2}$$

$$= \frac{i}{2} \ln \frac{r_2}{r_1} + \frac{\pi}{2} - \frac{i}{2}(\theta_2 - \theta_1)$$





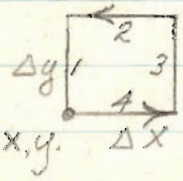
II Line Integrals



Consider the closed curve C. The line integral around C will be equal to the sum of the line integrals around the smaller regions into which we have divided the interior. This follows from the fact that all the integrals will cancel with equal and negative integrals except those along the boundary C.

Let us compute the integral around one of the small loops

$$\int f(z)\,dz$$



$$= \int (u\,dx - v\,dy) + i\int (v\,dx + u\,dy)$$

The real part:

$$\oint (u\,dx - v\,dy) = +v_1 \Delta y - u_2 \Delta x - v \Delta y + u_4 \Delta x$$

$$= -(v_3 - v_1)\Delta y - (u_2 - u_4)\Delta x$$

$$-\frac{\partial v}{\partial x}\Delta x \Delta y - \frac{\partial u}{\partial y}\Delta x \Delta y$$

$$= -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\Delta x \Delta y$$

The integral around the curve C

$$\int_C (u\,dx - v\,dy) \cong -\sum \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\Delta x \Delta y$$

$$= -\iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy$$

$$i\int_C (v\,dx + u\,dy) = \iint \left(-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right) dx\,dy$$

If these two integrals are to be zero

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

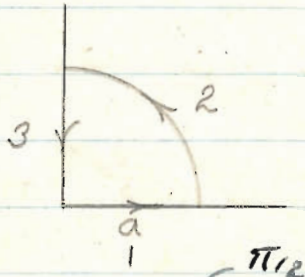
$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

These are simply the equations which hold if $f(z)$ is analytic:

The line integral of an analytic function around any closed curve is zero <u>provided</u> there are no singularities included by the curve.

Example of a Contour Integral:

$$W = \bar{z}$$



1) $\bar{z} = x \qquad \int_0^a x\,dx = \frac{a^2}{2}$

2) $\bar{z} = ae^{-i\theta} \qquad dz = iae^{i\theta}d\theta$

$$\int_0^{\pi/2} ae^{-i\theta}\, iae^{i\theta}\, d\theta = \frac{ia^2\pi}{2}$$

3) $\bar{z} = -iy \qquad \int_a^0 (-iy)\,i\,dy = -\frac{a^2}{2}$

$$\oint \bar{z}\,dz = i\frac{a^2\pi}{2}$$

$$= i\iint (1+1)\,dx\,dy = \frac{ia^2\pi}{2}$$

Problem: $W = z^2$

Math Methods

Lecture XII Nov 8 1946

Problems: By contour integration evaluate:

1) $\int_0^\infty \frac{dx}{1+x^4}$ 2) $\int_0^\infty \frac{x^2\,dx}{(a^2+x^2)^3}$

3) $\int_{-\infty}^{\infty} \frac{e^{iwt}}{w^2-w_0^2}\,dw$ below poles 4) $\int_0^\infty \frac{\sin x}{x(a^2+x^2)}$

5) $\int_0^{2\pi} \sin^2\theta\,d\theta/(a+b\cos\theta)$

6) $\int_0^{2\pi} d\theta/(a+b\cos\theta)^2$

7) $\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh \pi x}\,dx = \frac{1}{2}\tan\frac{a}{2}$

Hint: Integrate $\frac{e^{a^2}}{\sinh a\pi}$ on a rectangle (?)

8) Prove: $\int_0^\infty \frac{\cos x}{x+a}\,dx$

$$= \int_0^\infty \frac{x e^{-ax}}{1+x^2}\,dx$$

by integrating $\int \frac{e^{iz}}{z+a}$ on a square.

9) $\int_{-1}^{+1} dz / \sqrt{1-z^2}\,(a+bz) \quad a > b$

I Contour Integration

Let us integrate the function z^{-n} on a circle about the origin.

$$\oint \frac{1}{z^n}\,dz$$

$$z = a e^{i\theta}, \quad dz = i a e^{i\theta} d\theta$$

$$z^{-n} = a^{-n} e^{-in\theta}$$

$$\frac{i}{a^{n-1}} \int_0^{2\pi} e^{i\theta(1-n)}\,d\theta = \frac{1}{a^{n-1}} \frac{e^{i\theta(1-n)}}{(1-n)} \Big|_0^{2\pi}$$

$$= \frac{1}{a^{n-1}} \left(e^{i(1-n)2\pi} - 1\right)/(1-n)$$

This integral is zero for all integral values of n except $n = 1$. For $n = 1$, we have

$$i \int_0^{2\pi} d\theta = 2\pi i$$

$$\therefore \quad \oint_{\text{pole}} \frac{a}{z-b} = 2\pi i a$$

a is called the residue of the pole.

Suppose $f(z)$ has an isolated pole of order n at $z = b$. Let us study:

$$\oint_{\text{pole}} f(z)\,dz$$

The function $(z-b)^n f(z)$ has no pole at $z = b$. Expand in a Taylor series about b

$$(z-b)^n f(z) = a_{-n} + a_{-n+1}(z-b) + a_{-n+2}(z-b)^2 + \ldots$$

$$f(z) = \frac{a_{-n}}{(z-b)^n} + \frac{a_{-n+1}}{(z-b)^{n-1}} + \ldots$$

$$+ \frac{a_{-1}}{(z-b)} + a_0 + \ldots$$

We can integrate this series term by term. All the integrals will be zero except the one having the coefficient a_{-1}. The value

of this integral and hence of $\oint_{pole} f(z)\,dz$ is $2\pi i a_{-1}$

$$\oint_{pole} f(z)\,dz = 2\pi i a_{-1}$$

If n is the order of the pole a_{-1} is given by:

$$a_{-1} = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-b)^n f(z)\right]\right\}_{z=b}$$

For a pole of the first order

$$a_{-1} = (z-b) f(z)\Big|_{z=b}$$

If we write $f(z) = \frac{N(z)}{D(z)}$ where $D(b) = 0$

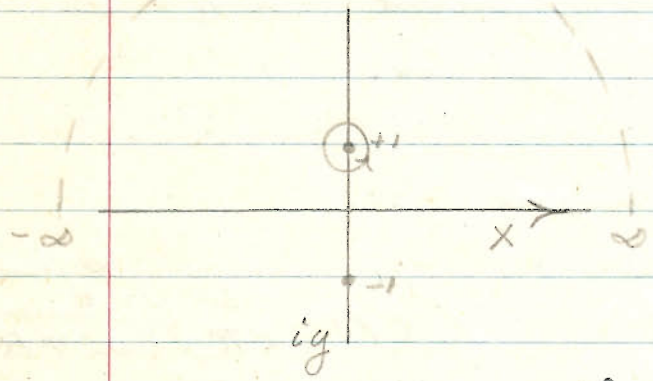
$$\lim_{z\to b} \frac{(z-b)N(z)}{D(z)} = \lim_{z\to 0} \frac{(z-b)N'(z) + N(z)}{D'(z)}$$

$$a_{-1} = N(b) / D'(b)$$

General Rules for Contour Integration:

Rule I Extend the range of integration as far as possible.

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$



$$\int_{-\infty}^{\infty} \frac{dz}{1+z^2}$$

$$\oint_{\text{pole}} \frac{dz}{1+z^2} = \int_{-\infty}^{\infty} \frac{dz}{1+z^2} + \int_{\text{circle}} \frac{dz}{1+z^2} = 2\pi i a_{-1}$$

real

$$\int_{\text{circle}}^{0} \frac{\pi i R e^{i\theta} d\theta}{1+R^2 e^{2i\theta}} \sim \frac{i}{R}\int_0^\pi e^{-i\theta} d\theta = 0$$

$R \to \infty$

$$= \frac{1}{1+z^2} \qquad D' = 2z$$

$$\frac{N(b)}{D'(b)} = \frac{1}{2z}\Big|_{z=i} = \frac{1}{2i}$$

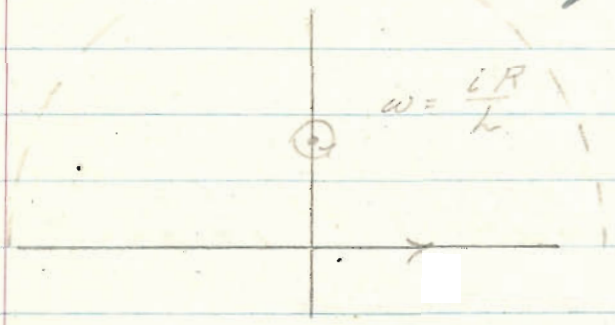
$$2\int_0^\infty \frac{dz}{1+z^2} = \int_{-\infty}^{\infty} \frac{dz}{1+z^2} = \pi$$

Contours enclosing more than one pole.



It is easy to see that the contour integral of a function around several poles will equal $2\pi i \Sigma$ residues at poles.

Another Example



$$I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{i\omega L + R}\, d\omega$$

$$\omega = \rho e^{i\theta}$$

$$\oint_{\text{pole}} = \int_{-\infty}^{\infty}_{\text{real}} + \int_{0}^{\pi}_{\text{circle}} d\theta\, i\rho e^{i\theta} e^{i\rho e^{i\theta} t} / (i\rho e^{i\theta} L + R)$$

$$\int_{0}^{\pi}_{\text{circle}} d\theta\, i\rho e^{i\theta} e^{i\rho(\cos\theta + i\sin\theta)t} / (i\rho e^{i\theta} L + R)$$

$$\int_{0}^{\pi}_{\text{circle}} d\theta\, e^{-\rho(\sin\theta)t} e^{i\rho(\cos\theta)t} / L \xrightarrow{\rho \to \infty}$$

For $t < 0$ integrate on lower.

$\int i$ circle $= 0$, no poles, hence

$$I(t) = 0 \qquad t < 0$$

For $t > 0$ integrate on upper contour. $\int i$ circle $= 0$

$$2\pi I(t) = 2\pi i \text{ Residue}$$

$$\text{Residue } \frac{e^{i\omega t}}{i\omega L + R} = \frac{e^{i\omega t}}{iL}\Big/_{\omega = \frac{iR}{L}}$$

$$I(t) = \frac{1}{L} e^{-\frac{R}{L}t}$$

Math Methods

Lecture XIII Nov 11, 1946

I Order of a Pole

The order of a pole may be determined by making the substitution:

$$z = b + \epsilon$$

and expanding. If the function goes as $1/\epsilon^n$ the pole is of order n.

The order of a pole may be determined by:

$$\lim_{z \to b} (z-b)^n f(z)$$

the value of n for which this expression no longer diverges is the order of the pole.

II Poles on the Contour

If there is a pole on a contour which is involved in the physical

problem, there is something wrong with the integral. A resistance effect may have been neglected. Try to determine what effect the introduction of such resistance might have on the position of the pole and then go above or below the pole as indicated.

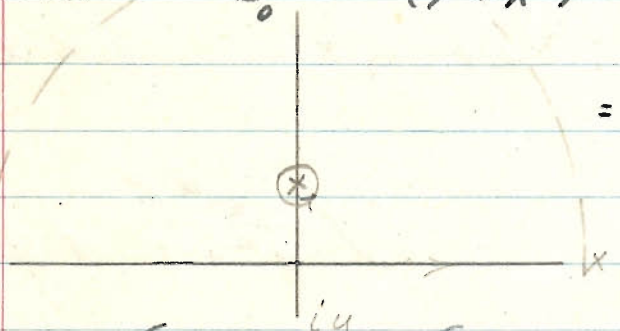
Math Methods

Lecture XIV Nov. 13, 1946

I Contour Integration: Examples

A. $\int_0^\infty \frac{\cos mx}{(1+x^2)^2}\,dx \quad m > 0$

$= R \int_{-\infty}^{\infty} \frac{e^{imz}}{(1+z^2)^2}\,dz$



$\oint_{\text{pole}} = \int_{\text{real}} + \int_{\frac{1}{2}\text{ circle}} = 2\pi i \text{ Residue}$

$\int_{\frac{1}{2}\text{ circle}} = 0$

The pole at $z = i$ is of order 2.

$$R = \frac{d}{dz}\left[(z-b)^2 f(z)\right]_{z=i}$$

$$= \frac{d}{dz}\frac{e^{imz}}{(z+i)^2}\Big|_{z=i}$$

$$= -\frac{i}{4}(m+1)e^{-m}$$

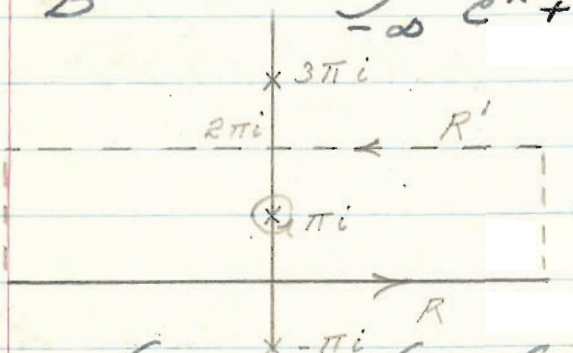
$$\int_0^\infty \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi}{4}(m+1)e^{-m}$$

$m > 0$

We observe the integral is symmetric in m. Therefore:

$$\int_0^\infty \frac{\cos mx}{(1+x^2)^2}\,dx = \frac{\pi}{4}(|m|+1)e^{-|m|}$$

B $$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx \qquad 0<a<1$$



$$\int_{pole} = \int_R + \int_{R'} + \int_{C=0} + \int_{C'=0} = 2\pi i R$$

$$\int_{R'} = \int_{\infty}^{-\infty} dx\, e^{a(x+2\pi i)}/(e^{x+2\pi i}+1)$$

$$= e^{2\pi a i} \int_{\infty}^{-\infty} \frac{e^{ax}}{e^x+1}\,dx$$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx = 2\pi i R/(1-e^{2\pi a i})$$

$$R = e^{az}/e^z\big|_{z=\pi i} = -e^{a\pi i}$$

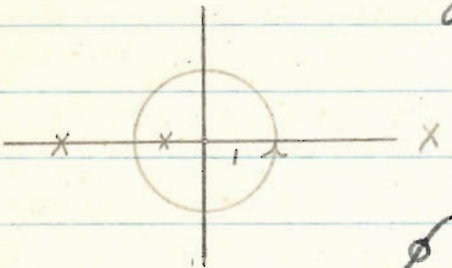
$$= -2\pi i\, e^{a\pi i}/(1-e^{2\pi a i})$$

$$= -2\pi i/(e^{-a\pi i}-e^{\pi a i})$$

$$= \pi / \sin \pi a \quad .$$

C. $$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$$

$a > b$



$$z = e^{i\theta}$$

$$dz = i e^{i\theta} d\theta$$

$$\oint_{\text{unit circle}} \frac{dz/iz}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} = 2\pi i R$$

$$= \frac{2}{i} \oint dz / (2az + bz^2 + b)$$

Poles for $z = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$

We note $z_1 z_2 = 1$ therefore one pole will be inside and one pole will be outside the unit circle.

$$R = \frac{2}{i} \, 1/(2a + 2bz)\Big|_{z = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}}$$

$$= 1/i\sqrt{a^2 - b^2}$$

$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \pi / \sqrt{a^2 - b^2}$$

Math Methods

Lecture XV Nov. 15, 1946

Problem: Find power series in t for

$$J_0(t) = \frac{1}{2\pi}\int_0^{2\pi} \cos(t\cos\theta)\,d\theta$$

I Residue at an Essential Singularity.

A function with an essential singularity may be expanded as follows:

$$f(z) = \ldots \frac{b_3}{z^3} + \frac{b_2}{z^2} + \frac{b_1}{z^1} + a_0 + a_1 z^1 + \ldots$$

It is obvious from this that the residue is simply b_1, the coefficient of $1/z$.

II Branch Points.

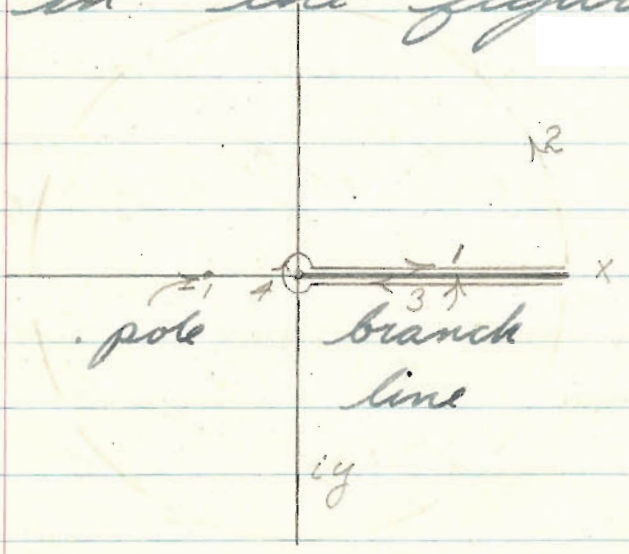
Consider the integral

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx \qquad 0 < p < 1$$

To find its value we investigate the complex integral:

$$\int \frac{z^{p-1}}{1+z}\,dz$$

The function $z^{p-1}/1+z$ has a branch point at the origin and a pole at $z=-1$. Let us attempt to integrate this function around the contour shown in the figure.



We avoid trouble with the branch line by simply not crossing it. Let choose our path on the surface for which $1^{p-1} = 1$

$$\int_{\text{contour}} = \int_1 + \int_2 + \int_3 + \int_4$$

$$\int_2 \to 0 \qquad \int_4 \to 0$$

$$\int_1 = \int_0^\infty \frac{x^{p-1}}{1+x}\,dx$$

x^{p-1} we mean the real positive root.

$$\int_3 = \int_\infty^0 \frac{x^{p-1} e^{2\pi i(p-1)}}{1+x}\,dx = -e^{2\pi i(p-1)} \int_0^\infty$$

Hence $\int_1 \cdot (1 - e^{2\pi i(p-1)}) = 2\pi i R$

The residue $R = (-1)^{p-1}$ The pole we enclose is the one on the surface already defined. The root we want is $e^{\pi i(p-1)}$

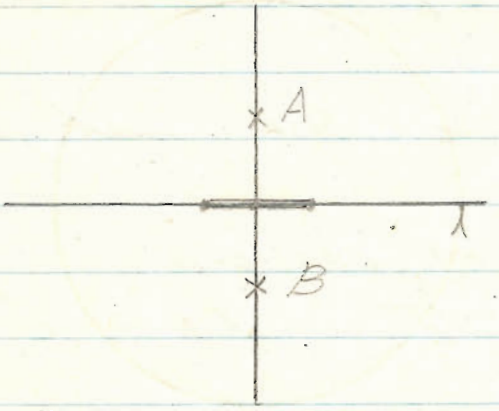
$$\int_1 = 2\pi i\, e^{\pi i(p-1)} / (1 - e^{2\pi i(p-1)})$$

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \pi / \sin p\pi$$

Let us compute the value of the line integral around a large circle of

$$\oint \frac{dz}{a + \sqrt{1 - z^2}}$$

circle ∞



Branch points at $z = \pm 1$
Poles at $z = \pm i \sqrt{a^2 - 1}$

Let us define our upper surface by saying that for large z $\sqrt{1 - z^2} \to +iz$

$$\oint \frac{dz}{iz} = \int_0^{2\pi} \frac{iRe^{i\theta} d\theta}{iRe^{i\theta}} = \int_0^{2\pi} d\theta = +2\pi$$

Let us now investigate which surface contains the poles. We come in from $+\infty$ on the y axis and see if we encounter a pole.

$$z = iy$$

$$\sqrt{1 - z^2} \pm \sqrt{1 + y^2}$$

We defined our upper surface

in such away that

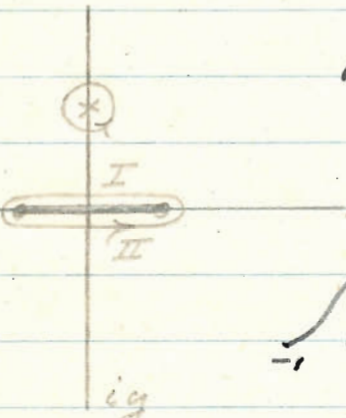
$$z \to \infty \qquad \sqrt{1-z^2} \to +iz$$

hence $\sqrt{1+y^2} \to +i(iy) = -y$

On the upper surface then take the root

$$-\sqrt{1+y^2}$$

The pole A is seen to be on the upper surface.



$$2\pi = 2\pi i R + \int_{I} + \int_{II}$$

$$\int_{-1}^{+1} \frac{dx}{a+\sqrt{1-x^2}} + \int_{+1}^{-1} \frac{dx}{a-\sqrt{1-x^2}}$$

$$\int_{-1}^{+1} \left(\frac{1}{a+\sqrt{1-x^2}} - \frac{1}{a-\sqrt{1-x^2}} \right) dx$$

$$Res \quad -\frac{\sqrt{1-z^2}}{z}\Big|_{z=i\sqrt{a^2-1}} = -\frac{-\sqrt{a^2}}{i\sqrt{a^2-1}}$$

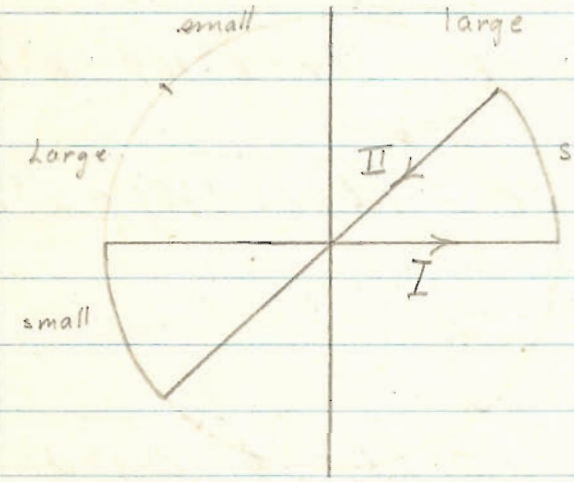
$$2\pi\left(1 - \frac{a}{\sqrt{a^2-1}}\right) = -\int_{-1}^{+1} \frac{2\sqrt{1-x^2}}{a^2-1+x^2}\, dx$$

Math Methods

Lecture XVI Nov 18, 1996

I Contour Integration

Consider the integral



$$\int e^{-z^2} dz$$

$$z = \rho e^{i\theta}$$

$$\int e^{-\rho^2 e^{2i\theta}} i\rho e^{i\theta} d\theta$$

$$\int_I + \int_{II} = 0$$

$$\int_I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$\int_{II} = -\int_{-\infty}^{\infty} e^{-i\rho^2} \frac{1}{\sqrt{2}}(1+i)\, d\rho$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \frac{1}{\sqrt{2}} \left\{ \int_{-\infty}^{\infty} (\cos\rho^2 - i\sin\rho^2)(1+i)\, d\rho \right.$$

$$= \frac{1}{\sqrt{2}} \left\{ \int_{-\infty}^{\infty} \cos\rho^2 d\rho + \int_{-\infty}^{\infty} \sin\rho^2 d\rho \right.$$

$$0 = -i\int_{-\infty}^{\infty} \sin\rho^2 d\rho + i\int_{-\infty}^{\infty} \cos\rho^2 d\rho$$

$$\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} \sin\rho^2 d\rho = \int_{-\infty}^{\infty} \cos\rho^2 d\rho$$

II Elliptic Integrals

A. First Kind

$$F(k,\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$= \int_0^{\sin\phi} dt \,/\, \sqrt{(1-t^2)(1-k^2t^2)}$$

$$= \int_0^{\sin^2\phi} dt \,/\, \sqrt{t(1-t)(1-k^2t)}$$

B. Second Kind

$$E(k,\phi) = \int_0^{\phi} d\theta \sqrt{1-k^2\sin^2\theta}$$

$$= \int_0^{\sin\phi} dt \sqrt{1-t^2} \,/\, \sqrt{1-k^2t^2}$$

C. Third Kind

$$\int_0^{\phi} d\theta \,/\, (1-n^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}$$

Integrals involving third or fourth power polynomials in roots are elliptic integrals. The first two kinds are tabulated.

D. Complete Elliptic Integrals (Tabulated)

First Kind

$$K(k) = \int_0^{\pi/2} d\theta \,/\, \sqrt{1 - k^2 \sin^2\theta}$$

Second Kind

$$E(k) = \int_0^{\pi/2} d\theta \, \sqrt{1 - k^2 \sin^2\theta}$$

E Some Functions

$$u = F(k\phi)$$
$$\phi = \mathrm{am}_k u$$

$$\sin\phi = \sin(\mathrm{am}_k u) = \mathrm{sn}\, u$$
$$\cos\phi = \mathrm{cn}\, u$$

$$\sqrt{1 - k^2 \sin\phi} = \sqrt{1 - k^2 \mathrm{sn}^2 u} = \mathrm{dn}\, u$$

III Doubly Periodic Functions

One real and one complex period

$$K\,;\; iK'$$

$$K' = K\sqrt{1 - k^2}$$

Math Methods

Lecture XVII Nov 20, 1946

I Contour Integration:

If one goes half way around a simple pole the value of the integral is πi times the residue.

II Elliptic Integrals

A Appearance in Physics

1 Period of Pendulum

2 Length of curves and conic sections

3 Inductance of coils

4 Solid angle problems: as when a collector looks at a target.

B Derivation

$$E(k) = \int_0^{\pi/2} d\theta \sqrt{1 - k^2 \sin^2\theta}$$

$$\frac{dE}{dk} = -\int_0^{\pi/2} d\theta \; k \sin^2\theta / \sqrt{1 - k^2 \sin^2\theta}$$

Note $$E = \int_0^{\pi/2} \frac{1 - k^2 \sin^2\theta}{\sqrt{1 - k^2 \sin^2\theta}} \, d\theta$$

$$E(k) = K(k) + k \frac{dE}{dk}$$

$$\frac{dE(k)}{dk} = \frac{1}{k}\left(E(k) - K(k)\right)$$

C Geo Arithmetic Mean

Arith M	1	→ 5	→ 4	→ 3.936
Geo M	9	→ 3	→ 3.873	→ 3.936

It can be shown that the geo arithmetic mean of m & n

$$\begin{matrix} m \\ n \end{matrix} = \pi/2 \int_0^{\pi/2} d\theta / \sqrt{m^2 \cos^2\theta + n^2 \sin^2\theta}$$

which can be reduce by the substitution $\cos^2\theta = 1 - \sin^2\theta$ to

$$G.A.\begin{pmatrix} m \\ n \end{pmatrix} = \pi m / 2 \, K(k)$$
$$k = 1 - \frac{n^2}{m^2}$$

70 Ref. Bartky RMP Vol 10

Math Methods

Lecture XVIII Nov. 22, 1946

I Gamma Function

Consider the integral

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$$

Take the n-1 derivative of both sides with respect to a

$$\int_0^\infty x^{n-1} e^{-ax}\,dx = \frac{(n-1)!}{a^n}$$

The above integral for $a = 1$ is defined as the Gamma Function of n, $\Gamma(n)$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx = (n-1)!$$

$$\frac{1}{a^n}\Gamma(n) = \int_0^\infty x^{n-1} e^{-ax}\,dx$$

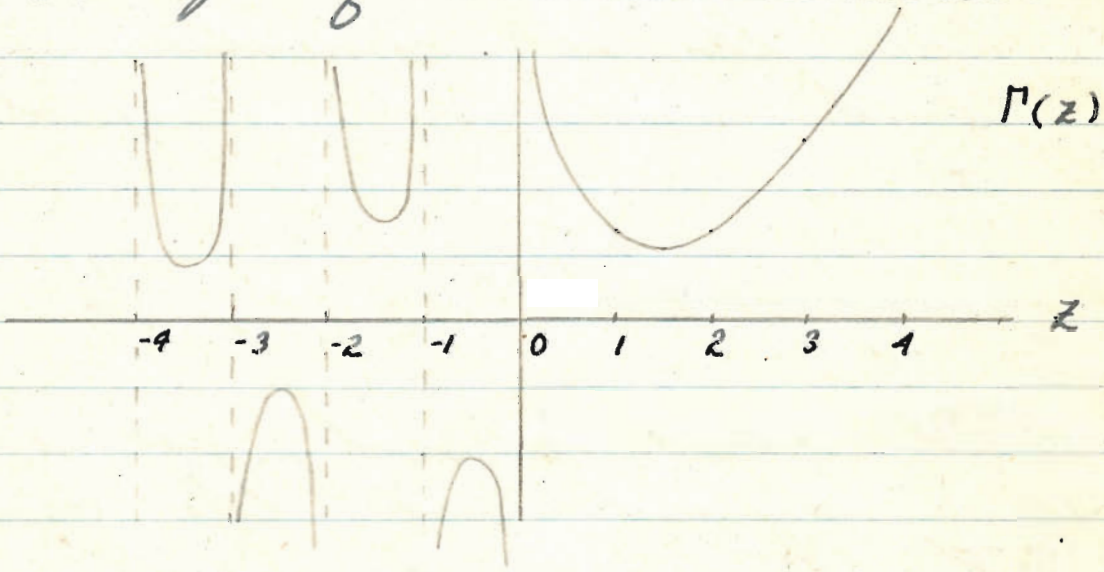
The Γ function is defined for integers greater than one only.

Recursion Formula

$$\Gamma(z+1) = z! = z(z-1)!$$
$$= z\,\Gamma(z)$$

$$\underline{\Gamma(z+1)} = \underline{z\,\Gamma(z)}$$

Graph of Gamma Function



Gamma of 1/2

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\,dx$$

$x = y^2$

$$= \int_0^\infty \frac{1}{y} e^{-y^2} 2y\,dy$$

$$= 2\int_0^\infty e^{-y^2} dy = \sqrt{\pi}$$

$$\Gamma(1\tfrac{1}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}$$

Beta Function

$$\int_0^1 (1-x)^{m-1} x^{n-1} dx \equiv B(m,n)$$

$$B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$$

Proof: $\Gamma(m)\Gamma(n)$

$$= \int y^{m-1} e^{-y} dy \int x^{m-1} e^{-x} dx$$

$$y = xt, \quad dy = x\,dt, \quad dx = dx$$

$$= \int_{x=0}^{\infty} \int_{t=0}^{\infty} (xt)^{m-1} e^{-xt} e^{-x} x^{n-1} x\,dx\,dt$$

$$= \int_0^{\infty} dt\, t^{m-1} \int_0^{\infty} x^{(m+n-1)} e^{-(1+t)x} dx$$

$$= \int_0^{\infty} dt \left[t^{m-1}/(t+1)^{m+n}\right] \Gamma(m+n)$$

$$\Gamma(m)\Gamma(n)/\Gamma(m+n) = \int_0^{\infty} dt\, t^{m-1}/(t+1)^{m+n}$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1} dx$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta$$

$$= B(m,n)$$

Consider the special case

$$m + n = 1$$

$$\Gamma(m)\Gamma(1-m) = \int_0^\infty \frac{t^{m-1}}{(1+t)}\, dt$$

$$= \pi / \sin m\pi$$

$$\underline{\Gamma(z)\Gamma(1-z) = \pi / \sin \pi z}$$

Problem:

Consider $I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$

Show by two methods

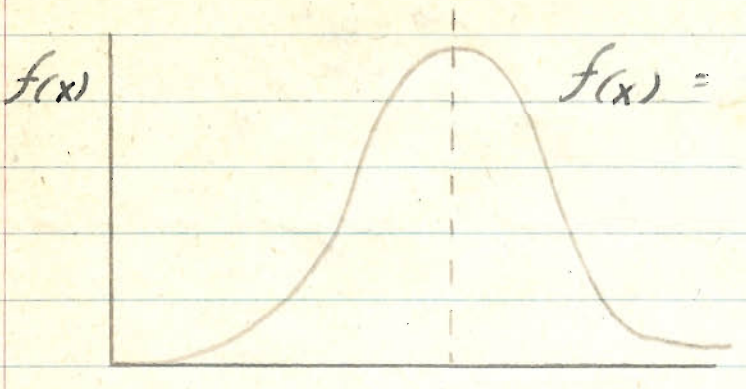
$$I = \sqrt{2}\, K\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\sqrt{\pi}\, \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$$

Find $\Gamma(\frac{1}{4})$ in terms of elliptic integral.

II Stirling Approximation for the Γ function.

For large n

$$n! = \int_0^\infty x^n e^{-x}\, dx$$



$f(x) = x^n e^{-x}$

Let us expand this function around its maximum.

$$f = e^{-x + n \ln x}$$

$$\frac{df}{dx} = -1 + n/x = 0$$

Maximum at $n = x$

Let $y = x - n$

Expand the exponent

$$-x + n \ln x = -n - y + n \ln(n + y)$$
$$= -n - y + n \ln n + n \ln\left(1 + \frac{y}{n}\right)$$
$$= -n - y + n \ln n + n \frac{y}{n} - \frac{n}{2} \frac{y^2}{n^2} +$$

$$\cong -n + n \ln n - \frac{y^2}{2n}$$

$$\int_0^\infty x^n e^{-x} dx \cong \int_{-n}^\infty e^{-n + n \ln n - \frac{y^2}{2n}} dy$$

$$= n^n e^{-n} \int_{-n}^{\infty} e^{-\frac{y^2}{2n}} dy$$

For large n replace $-n$ by $-\infty$.

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

A better approximation is

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n}\right)$$

n	$n!$	Ster.	Ster. Cor.
3	6	5.86	6.00
2	2	1.92	2.00
1	1	.923	.998
$1\frac{1}{2}$	$.5\sqrt{\pi}$	$.71\sqrt{\pi}$	$.83\sqrt{\pi}$

As n becomes large the absolute error becomes large but the percentage becomes smaller.

Math Methods

Lecture XIX Nov. 25, 1996

I Discussion of Problems on Contour Integration.

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(t\cos\theta)\, d\theta$$

$$z = e^{i\theta},\quad d\theta = \frac{dz}{iz}$$

$$\frac{1}{2\pi}\oint \cos\left(\frac{t}{2}\left[z+\frac{1}{z}\right]\right)\frac{dz}{iz}$$

$$\frac{1}{2\pi i}\oint \frac{1}{z}\, e^{\frac{it}{2}\left(z+\frac{1}{z}\right)}\, dz$$

Pole at $z = 0$ (essential)
The value of the integral is $2\pi i$ times the residue at the pole. The residue is the coefficient of $\frac{1}{z}$ in the expansion of the function

$$\frac{1}{z}\, e^{\frac{itz}{2}}\, e^{\frac{it}{2z}}$$

$$= \frac{1}{z}\sum_m \frac{\left(\frac{itz}{2}\right)^m}{m!}\sum_n \frac{\left(\frac{it}{2z}\right)^n}{n!}$$

$$\frac{1}{z}\sum_{m,n}\frac{1}{m!n!}\left(\frac{it}{2}\right)^{m+n}z^{m-n}$$

We have the coefficient of $\frac{1}{z}$ for $m = n$

$$\mathcal{R} = \frac{1}{2\pi i}\sum_{n}\frac{\left(i\frac{t}{2}\right)^{2n}}{(n!)^2}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(t\cos\theta)\,d\theta = \sum_n \frac{1}{(n!)^2}\left(-\frac{t^2}{4}\right)^n$$

Math Methods

Lecture XX Nov 27, 1946

I Cauchy Integral Theorem.

Suppose $f(z)$ analytic, then

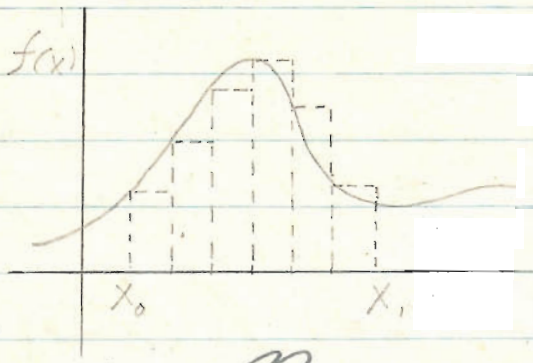
$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)}\, dz$$

$$f^n(a) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}}\, dz$$

The first of the above equations tells us that an analytic function can have no maximum inside the closed contour.

The fact that we know the function and all its derivatives at a enable us to expand it in a Taylor Series about a. Hence, if the value of the function is known at the boundary its value at any point inside may be determined.

II Numerical Solution of Integrals

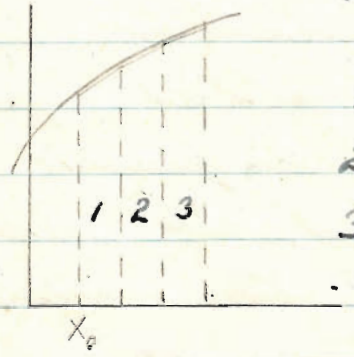


A. Riemann Sum

Divide the interval of integration into n interval of width, h,

$$\int_{X_0}^{X_1} f(x)\,dx \sim \sum_{n'=0}^{n-1} h\, f(X_0 + n'h)$$

B. Trapezoidal Rule



$$1 = \frac{h}{2}\left(f[x] + f[X_0 + h]\right)$$
$$2 = \frac{h}{2}\left(f[X_0 + h] + f[X_0 + 2h]\right)$$
$$3 = \frac{h}{2}\left(f[X_0 + 2h] + f[X_0 + 3h]\right)$$
$$\ldots = \ldots$$

We shall employ the following shorthand to indicate how one should proceed in taking

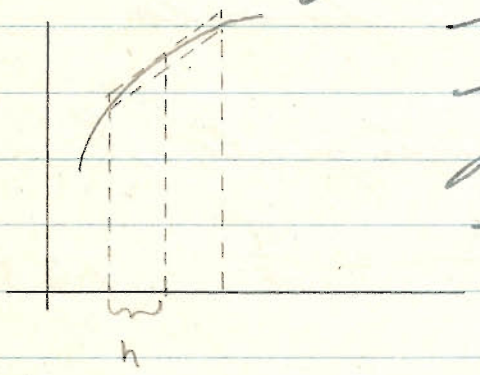
a sum

$$h\left[\tfrac{1}{2}f(x_0) + 1f(x_0+h) + 1f(x_0+2h) + \ldots\right]$$

$$= h\left(\tfrac{1}{2}\ 1\ 1\ 1 \ldots 1\ \tfrac{1}{2}\right) \to h\left(\tfrac{1}{2}\ \tfrac{1}{2}\right)$$

C. Simpson's Rule

Fit a parabola thru three adjacent points. The area under a section of a parabola included in a rectangle is 2/3 the area of the rectangle



$$A = \frac{h}{3}(1\ 4\ 1)$$

$$= \frac{h}{3}(1\ 4\ 2\ 4\ 2\ 4\ 1)$$

It is obvious that in order to employ Simpson's Rule we must divide the interval of integration into an even number of spaces.

B. $3/8$ Rule

Fit sections of a cubic to the curve.

$$A = \frac{3h}{8}(1\,3\,3\,1)$$

$$= \frac{3h}{8}(1\,3\,3\,2\,3\,3\,1)$$

To use $3/8$ Rule one must have $3n$ intervals.

Example: Integrate numerically

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} = .78540$$

x	$f(x)$	Trap.		Simp	
0	1.00000	$\frac{1}{2}$	.50000	1	1.00000
.25	.94117	1	.94117	4	3.76470
.50	.80000	1	.80000	2	1.60000
.75	.64000	1	.64000	4	2.56000
1.0	.50000	$\frac{1}{2}$	.25000	1	.50000

$4)\underline{3.1312}$ $12)\underline{9.42470}$

$\frac{\pi}{4} = .78540$ $.7828$ $.78538$

Math Methods

Home Problem

Integrate by Simpsons Rule

$$\int_0^1 x^4 \, dx$$

$$\int_0^{2\pi} d\theta / (1 + \sin^2\theta) = \pi\sqrt{2}$$

Table of $\int_0^x \frac{\sin x}{x} \, dx$

For $x = 0$ to π in steps of $\pi/6$.

Math Methods
Lecture XXI Nov 29, 1946

I Numerical Integration

Computation of a running integral

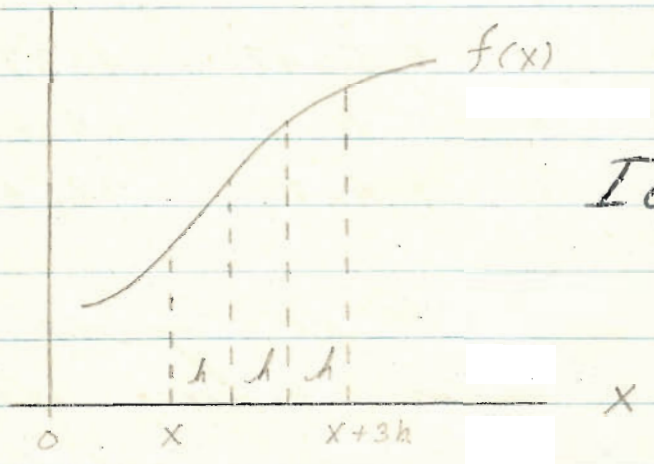
$$f(x) = \int_0^x \frac{dt}{1+t^2}$$

x	$\frac{1}{1+t^2}$	SR	$12\int_0^x$	$\int_0^x$	exact
0	1.0	1	0.0	0.0	0.0
.25	.9412	4	a	.2452	.2450
.50	.8000	1·1	5.5648	.4637	.4637
.75	.6400	4	b	.6436	.6435
1.0	.5000	1	9.4248	.7854	.7854

Simpsons Rule gives the integral only every other interval. If the value of either a or b is known Simpsons Rule may be used to compute the missing values from there on. a may be found by subdividing the first interval and using Simpsons Rule; b may be found by using the

3/8 Rule. The 3/8 Rule is not as exact as Simpsons Rule and should be avoided where possible. Computation of alternate terms separately provides a check on the work since at any point the answers should agree.

Derivation of 3/8 Rule.



$$I(x) = \int_0^x f(x)\,dx$$

Write $I(x+3h) = I(x) + a f(x) + b f(x+h) + c f(x+2h) + b f(x+3h)$. To find the coefficients a, b, c, & d, expand both sides in a Taylor Series and equate terms in like derivatives.

$$I(x+3h) = I(x) + af(x) + bf(x+h) + cf(x+2h) + df(x+3h)$$

$$I(x) = I(x)$$

$$3hI' = +af + bf + cf + df$$

$$\frac{9h^2}{2}I'' = +bhf' + c2hf' + d3hf'$$

$$\frac{27h^3}{6}I''' = +\frac{bh^2}{2}f'' + \frac{c4h^2}{2}f'' + \frac{d9h^2}{2}f''$$

$$\frac{81h^4}{24}I^{IV} = +\frac{bh^3}{6}f''' + \frac{c8h^3}{6}f''' + \frac{d27h^3}{6}f'''$$

$$\frac{243h^5}{120}I^{V} = +\frac{bh^4}{24}f^{IV} + \frac{c16h^4}{24}f^{IV} + \frac{d81h^4}{24}f^{IV}$$

$$+ \text{error}$$

Equating the coefficients of like derivatives:

$$\begin{aligned} a + b + c + d &= 3h \\ b + 2c + 3d &= 9h/2 \\ b + 4c + 9d &= 9h \\ b + 8c + 27d &= 81h/4 \end{aligned}$$

$$\text{Error} = \left(\frac{243}{120} - \frac{b + 16c + 81d}{24h}\right) h^5 f^{IV}$$

A solution of the above equations yields

$$a = \frac{3}{8}$$
$$b = \frac{9}{8}$$
$$c = \frac{9}{8}$$
$$d = \frac{3}{8}$$

Hence $I(x+3h) - I(x) = \frac{3}{8} h (1\ 3\ 3\ 1)$

$$\text{Error} \sim -\frac{3}{80} h^5 f^{IV}$$
$$\text{Error / Interval} \sim -\frac{1}{80} h^5 f^{IV}$$

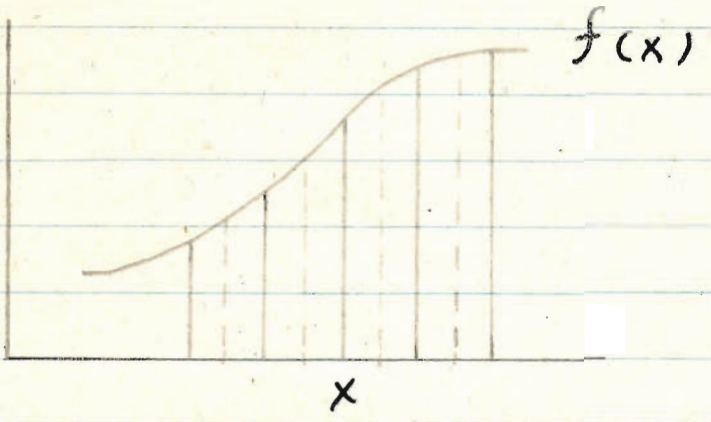
For Simpsons Rule

$$\text{Error} \sim \pm \frac{1}{180} h^5 f^{IV}$$

The fourth derivative may be approximated by taking the fourth difference of the tabulated values. Taking the differences provides a check on the smoothness of the function.

Problem:

I



Prove:

$$f(x) = \frac{9}{16}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$- \frac{1}{16}\left[f\left(x+\frac{3h}{2}\right) + f\left(x-\frac{3h}{2}\right)\right]$$

II Find by Simpsons Rule

$$\int_0^1 2^x \, dx$$

Math Methods

Lecture XXII Dec. 3, 1946

I Simpson Rule

A. Total error in the interval x_1 to x_2

$$\text{Error} \sim \sum_i \frac{h^5}{180} f_i^{IV}$$

$$\sim \int_{x_1}^{x_2} \frac{h^5}{180} f^{IV} \frac{dx}{h}$$

$$= \frac{h^4}{180} \left[f^{III}(x_2) - f^{III}(x_1) \right]$$

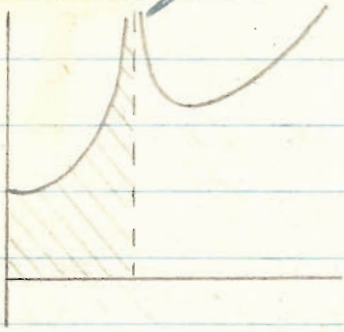
B Integrals for which Simpsons Rule does not work:

1) No good for integrals over an infinite range.

2) No good for integrals in which the integrand goes to infinity.

3) No good for integrals in which the integrand goes to zero as a root.

If an integral possesses property 2 or 3 above, try to evaluate the integral in the neighborhood of the singularity by approximate analytical treatment. For example consider:



$$\int_0^1 \frac{\cosh x^2}{\sqrt{1-x}}\,dx$$

This integrand goes to infinity as $x \to 1$. Therefore, evaluate the integral from 0 to .9 using Simpson Rule and from .9 to 1 as follows:

$$\int_{.9}^{1} \frac{\cosh x^2}{\sqrt{1-x}}\,dx \sim \int_{.9}^{1} \frac{\cosh 1}{\sqrt{1-x}}\,dx$$

$$= -\cosh 1 \, 2(1-x)^{\frac{1}{2}} \Big|_{.9}^{1}$$

$$= 1.52 \cdot 2\sqrt{.1}$$

If the above value for the integral is not good

enough expand the cosh x^2 and keep the next term in the series.

Integrals over an infinite range can sometimes be transformed into integrals over a finite range or into integrals in which the integrand goes rapidly to zero for large values of the argument. For example:

$$\int_0^\infty \frac{\sin mt}{1+t^2}\,dt = e^{-m}\int_0^m \frac{e^x}{x}\,dx + e^{m}\int_m^\infty \frac{e^{-x}}{x}\,dx$$

The integrals on the RHS can be evaluated by Simpsons Rule.

II Tabulated Integrals

A. Exponential Integral

$$-Ei(-x) = \int_x^\infty \frac{e^{-y}}{y}\,dy$$

$$\bar{E}i(x) = \int_{-\infty}^{x} \frac{e^y}{y}\, dy$$

B. Error Integral

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

C. Bessel Integral

$$J_n(v) = \frac{1}{\pi} \int_0^{\pi} \cos(nt - v\sin t)\, dt$$

D. Cosine Integral

$$Ci(x) = -\int_x^{\infty} \frac{\cos t}{t}\, dt$$

Sine Integral

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt$$

$$\frac{d\,Si(x)}{dx} = \frac{\sin x}{x}$$

Let us investigate first the exponential integral.

A. For small x

$$-Ei(-\epsilon) = \int_{\epsilon}^{\infty} \frac{e^{-y}}{y}\, dy = C - \ln \epsilon$$

Solving for C

$$C \equiv \int_{\epsilon}^{\infty} \frac{e^{-y}}{y}\,dy + \ln \epsilon$$

$$= \int_{\epsilon}^{\infty} \frac{e^{-y}}{y}\,dy - \int_{\epsilon}^{1} \frac{1}{y}\,dy$$

$$= \int_{1}^{\infty} \frac{e^{-y}}{y}\,dy + \int_{\epsilon}^{1} \frac{e^{-y}-1}{y}\,dy$$

$$C \simeq \int_{1}^{\infty} \frac{e^{-y}}{y}\,dy + \int_{0}^{1} \frac{e^{-y}-1}{y}\,dy$$

→ Problem: Find C numerically to 3 figures.

→ Show: $-C = -\int_{0}^{\infty} \ln y\, e^{-y}\,dy$

→ Show: $-Ei(-\epsilon) \sim -\ln(1.781\epsilon)$

Show:

$$Ci(\epsilon) = -\int_{\epsilon}^{\infty} \frac{\cos t}{t}\,dt \doteq -\ln(1.781\epsilon)$$

B. For large x

$$-Ei(-m) = \int_{m}^{\infty} \frac{e^{-y}}{y}\,dy$$

$$\sim \frac{1}{m}\int_{m}^{\infty} e^{-y}\,dy = \frac{e^{-m}}{m}$$

Let us try integration by parts on the error integral.

$$\int_x^\infty \frac{e^{-y}}{y}\,dy = -\frac{1}{y}e^{-y}\Big|_x^\infty - \int_x^\infty \frac{e^{-y}}{y^2}\,dy$$

$$= -\frac{1}{y}e^{-y}\Big|_x^\infty + \frac{1}{y^2}e^{-y}\Big|_x^\infty + 2\int_x^\infty \frac{e^{-y}}{y^3}\,dy$$

$$= \frac{1}{x}e^{-x} - \frac{1!}{x^2}e^{-x} + \frac{2!}{x^3}e^{-x} - \dots$$

$$-Ei(-x) = e^{-x}\left[\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots\right]$$

This series for $-Ei(-x)$ is an Asymtotic Series. It has the following properties:

1) It diverges

2) The terms alternate in sign.

3) The error made up to and including n terms is less than the n^{th} term.

We can see that the accuracy available with such a series depends on the value of x. For large x it is very good.

Math Methods

I Exponential Integral

$$\int_0^\infty \frac{e^{-x}}{x+a}\,dx = -e^a Ei(-a)$$

$$\int_0^\infty \frac{e^{-tx}}{t}\,dx \quad (u=tx) \quad = \int_x^\infty \frac{e^{-u}}{u}\,du = -Ei(-u)$$

$$\int_0^y Ei(-x)\,dx = x\,Ei(-x)\Big|_0^y - \int_0^y x\,\frac{e^{-x}}{x}\,dx$$

$$= y\,Ei(-y) + e^{-y} - 1$$

Problem (Home) Find $\int_y^\infty \frac{e^{-x}}{x^3}\,dx$

II Sine Integral $Si(x) = \int_0^x \frac{\sin t}{t}\,dt$

Consider:

$$f(pq) = \int_0^\infty e^{-pt} Si(qt)\,dt$$

$$= -\frac{1}{p} e^{-pt} Si(qt)\Big|_0^\infty + \frac{1}{p}\int_0^\infty e^{-pt}\frac{\sin qt}{t}\,dt$$

$$\frac{\partial(pf)}{\partial t} = -\int e^{-pt}\sin qt\,dt = -\frac{q}{p^2+q^2}$$

$$pf = -\int \frac{q\,dp}{p^2+q^2} = -\tan^{-1}\frac{p}{q} + C$$

For $q = 0$ $f = 0$ hence:

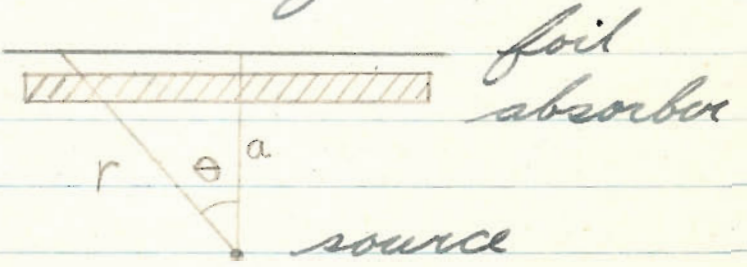
$$f = \frac{1}{p}\left(\frac{\pi}{2} - \tan^{-1}\frac{p}{q}\right) = \frac{1}{p}\tan^{-1}\frac{q}{p}$$

Problem (Home) Find:

$$\int_x^\infty \frac{-Ei(-x-u)}{u}\,du$$

$$\int_0^\infty \frac{\sin mt}{1+t^2}\,dt = \frac{1}{2}\left[e^{-m}Ei(m) - e^{m}Ei(-m)\right]$$

III A Physical Problem:
Absorbtion of Neutrons by a thin foil.



The number of neutrons falling on a square centimeter of the foil from a unit source will be:

$$N_0 = \frac{1}{r^2}\cos\theta$$

The probability that the neutron is captured by the

foil is proportional to length of the neutrons path through the foil

$$\sigma_c \sim t / \cos\theta$$

If an absorber is now placed between the source and the foil the number of neutrons reaching the foil will be reduced. The factor by which the number of neutrons is reduced is:

$$\frac{N}{N_0} = e^{-y/\lambda}$$

λ is the mean free path of a particle in the absorber and y is the distance in the absorber that the particles have traversed.

11 Using the above information and the following data compute λ.

a = 12 cm
radius of foil = 12 cm
thickness of absorber = 1 cm
reduction in number of
counts from foil .38 X

21 Consider an infinitely large foil and a source close to the absorber. Assume the error in measurement of the activity $\sim\sqrt{\text{activity}}$. $\frac{\Delta N}{N} \sim \frac{1}{\sqrt{N}}$

How thick should the absorber be to permit the most accurate determination of λ. Suppose at the optimum thickness the activity can be determined to $\pm 1\%$. What will be the accuracy in λ.

(Correction:)
Initially a standard absorber is placed between source and foil. Its thickness is equal to its mean free path λ.

II Error Function W.P.A.

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Series evaluation

$$\int_0^x e^{-t^2} dt = x - \frac{x^3}{1!3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + \dots$$

Converges for all x, but slowly for large x. For large x we can find an asymtotic series:

$$\frac{\sqrt{\pi}}{2}(1 - Erf(x)) = \int_x^\infty e^{-t^2} dt = \int_x^\infty \frac{e^{-t^2}}{t} t\,dt$$

$$= -\frac{e^{-t^2}}{2t}\Big|_x^\infty - \int_x^\infty \frac{e^{-t^2}}{2} \frac{dt}{t^2}$$

$$= \frac{e^{-x^2}}{2x} - \int_x^\infty \frac{e^{-t^2}}{2t^2} dt$$

$$= \frac{e^{-x^2}}{x}\left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} + \dots\right)$$

$$= \frac{\sqrt{\pi}}{2}[1 - Erf(x)]$$

Math Methods

Lecture XXIV Dec. 6, 1946

I Eulers Constant

$$-C = .577 = -\int_0^\infty e^{-x} \ln x \, dx$$

$$C = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right)$$

II Error Function

$$Erf(x) = \int_0^x e^{-y^2} dy$$

$$\int_0^x Erf(x)\, dx = \frac{1}{2}(1 - e^{-x^2}) + x\, Erf(x)$$

$$\int_{-\infty}^{\infty} e^{i\omega x - a^2 x^2} dx = \sqrt{\pi}\, e^{-\frac{\omega^2}{4a^2}}$$

$$\int_{-\infty}^{\infty} \frac{\sqrt{\pi}}{a} e^{-\frac{(x-x')^2}{a^2}} \cdot \frac{\sqrt{\pi}}{a} e^{-\frac{x'^2}{b^2}} dx'$$

$$= \frac{\sqrt{\pi}}{\sqrt{a^2+b^2}}\, e^{-\frac{x^2}{a^2+b^2}}$$

III Heat Flow:

$$\frac{\partial Q}{\partial t} = K \frac{\partial T}{\partial x}$$

$$c \frac{\partial T}{\partial t} \Delta x = \frac{\partial}{\partial x} \frac{\partial q}{\partial t} \Delta x$$

$$\frac{\partial T}{\partial t} = \frac{1}{c} \frac{\partial}{\partial x} K \frac{\partial T}{\partial x}$$

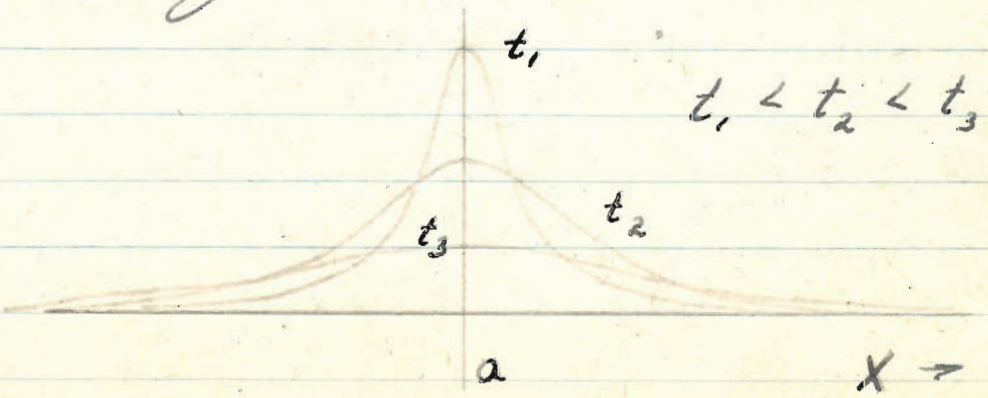
If K & c independent of x and x & t

$$\frac{\partial T}{\partial t} = \frac{K}{c} \frac{\partial^2 T}{\partial x^2}$$

It is easily shown that

$$T(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-a)^2}{4t}}$$

is a particular solution of the above differential equation. At various times this function has the form shown below.



The area under the temperature curves is constant. This is equivalent to saying that the total quantity of heat is conserved.

$$\int_{-\infty}^{\infty} T(x\,t)\,dx = \text{Constant}$$

$$T(x\,0) = \delta(x-a)$$

If T is a solution of the heat equation, then

$$T_1(x\,t) = \sum_i \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-a_i)^2}{4t}} c_i$$

is also a solution. And, in general,

$$T(x\,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(a)\, e^{-\frac{(x-a)^2}{4t}}\, da$$

is a solution for which

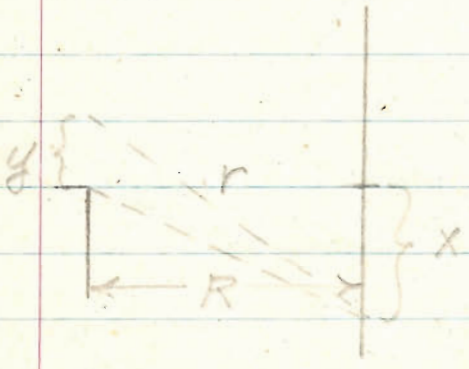
$$T(x\,0) = f(x)$$

Problem: for the following initial temperature distribution

$$T(x0) = I(x)$$

find $T(xt)$

Problem: (Hand in)
Compute Diffraction pattern at a knife edge.



If $\lambda << R$, the amplitude at X

$$A(xt) \cong A_0 \int_0^\infty \cos\omega\left(t - \frac{r}{c}\right) dy$$

$$r^2 = R^2 + (x+y)^2$$

If $\lambda << R$ x and $y << R$

$$r \sim R + \frac{1}{2R}(x+y)^2$$

Average over a period of A_x^2 equal average intensity

$$I = \int_0^{\frac{2\pi}{\omega}} [A(xt)]^2 dt$$

Fresnel Integrals (Tabulated)

$$\int_0^u \cos(x^2)\,dx$$

$$\int_0^u \sin(x^2)\,dx = \int_0^u \frac{\sin t}{\sqrt{t}}\,dt$$

There are tables of

$$\int e^{+x^2}\,dx \quad \text{Jahnke-Emde}$$

Math Methods

Lecture XXV Dec. 9, 1946

I Bessel Function

$$J_n(v) = \frac{1}{\pi}\int_0^{\pi} \cos(nt - v\sin t)\,dt$$

$$= \frac{1}{2\pi i^n}\int_0^{2\pi} e^{iv\cos t}\cos(nt)\,dt$$

These definitions hold for integral n.

We shall now prove some interesting properties of the Bessel Function

Consider:

$$\int_0^{\pi}(n - v\cos t)\cos(nt - v\sin t)\,dt$$

$$= \frac{1}{\pi}\sin(nt - v\sin t)\Big|_0^{\pi} = 0$$

Hence

$$n J_n(v) = \frac{v}{\pi}\int_0^{\pi}\cos t\cos(nt - v\sin t)\,dt$$

Using the relation $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$= \frac{v}{2\pi}\int_0^{\pi}\cos[(n+1)t - v\sin t]\,dt$$

$$+ \frac{v}{2\pi}\int_0^{\pi}\cos[(n-1)t - v\sin t]\,dt$$

1 $n J_n(v) = \frac{v}{2}\left[J_{n+1}(v) + J_{n-1}(v) \right]$

Consider:

$$\frac{d}{dv}\left[J_n(v) \right] = J_n'(v)$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin t \sin(nt - v\sin t)\, dt$$

Using $2 \sin A \sin B = \cos(A+B) - \cos(A-B)$

2 $J_n'(v) = \frac{1}{2}\left[J_{n-1}(v) - J_{n+1}(v) \right]$

The following relations can be easily shown to hold:

3 $J_n' = J_{n-1} - \frac{n}{v} J_n$

4 $J_n' = \frac{n}{v} J_n - J_{n+1}$

5 $\frac{d}{dv} v^n J_n = v^n J_{n-1}$

6 $\frac{d}{dv} v^{-n} J_n = -v^{-n} J_{n+1}$

7 $\int_0^v J_1\, dv = -J_0 + 1$

8 $$\int_0^v J_0\, v\, dv = v\, J_1$$

9 $$\int_0^v [J_0]^2 v\, dv = \frac{v^2}{2}[J_0^2 + J_1^2]$$

10 $$\int_0^v [J_n]^2 v\, dv = \frac{v^2}{2}[J_n^2 - J_{n-1} J_{n+1}]$$

11 $$\int_0^v J_0\, dv = \sum_{i=1}^{\infty} J_{2i-1}$$

The Bessel Equation

12 $$J_n'' + \frac{1}{v} J_n' + \left(1 - \frac{n^2}{v^2}\right) J_n = 0$$

We shall show that our function satisfies this equation.

From 2

$$J_n'' = -\frac{n}{v^2} J_n + \frac{n}{v} J_n' - J_{n+1}'$$

From 3 $J_{n+1}' = J_n - \frac{n+1}{v} J_{n+1}$

From 2 $J_{n+1} = -J_n' + \frac{n}{v} J_n$

$$-J_n'' - \frac{n}{v^2} J_n + \frac{n}{v} J_n' - J_n + \frac{n+1}{v}\left(\frac{n}{v} J_n - J_n'\right) = 0$$

$$J_n'' + \frac{1}{v} J_n' + \left(1 - \frac{n^2}{v^2}\right) J_n = 0$$

Problem: Show

$$\int_0^\infty \frac{1}{x} J_0(xt) \sin x \, dx$$

$$= \frac{\pi}{2} \qquad 0 < t < 1$$

$$= \csc^{-1} t \qquad t > 1$$

Hint: Try $t = \csc u$; Contour integrate.

Power Series for $J_0(v)$

$$J_0(v) = \frac{1}{\pi} \int_0^\pi \cos(v \sin t) \, dt$$

$$= \frac{1}{\pi} \sum_0^\infty \frac{(-1)^k}{(2k)!} v^{2k} \int_0^\pi \sin^{2k} t \, dt$$

But $\Gamma(m)\Gamma(n) / \Gamma(m+n)$

$$= \int_0^\pi \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$n = \tfrac{1}{2} \; ; \; 2k = 2m - 1$$

$$\Gamma(k + \tfrac{1}{2}) \Gamma(\tfrac{1}{2}) / \Gamma(n+1) = \int_0^\pi \sin^{2k}\theta \, d\theta$$

$$J_0(v) = \frac{1}{\pi} \sum_0^\infty \frac{(-1)^k v^{2k} \sqrt{\pi}}{(2k)! \, k!} \Gamma(k + \tfrac{1}{2})$$

$$\Gamma(k+\tfrac{1}{2}) = (k-\tfrac{1}{2})(k-\tfrac{3}{2})\cdots(\tfrac{1}{2})\Gamma(\tfrac{1}{2})$$

$$= \frac{1}{2^k}\left[1\cdot 3\cdot 5\cdot 7\cdots(2k-1)\right]\sqrt{\pi}$$

$$= \left(\frac{\sqrt{\pi}}{2^k}\right)(2k)! \,/\, 2\cdot 4\cdot 6\cdot 8\cdots 2k$$

$$= \sqrt{\pi}(2k)! \,/\, 4^k\, k!$$

$$J_0(v) = \sum_0^\infty \frac{(-1)^k}{(k!)^2}\left(\frac{v}{2}\right)^{2k}$$

Math Methods

Lecture XXVI Dec. 11, 1946

I Bessel Function (cont.)

$$J_0(v) = \sum_0^\infty \frac{(-1)^k}{(k!)^2}\left(\frac{v}{2}\right)^{2k}$$

$$J_n(v) = \frac{v}{2}\left[J_{n+1} + J_{n-1} \right]$$

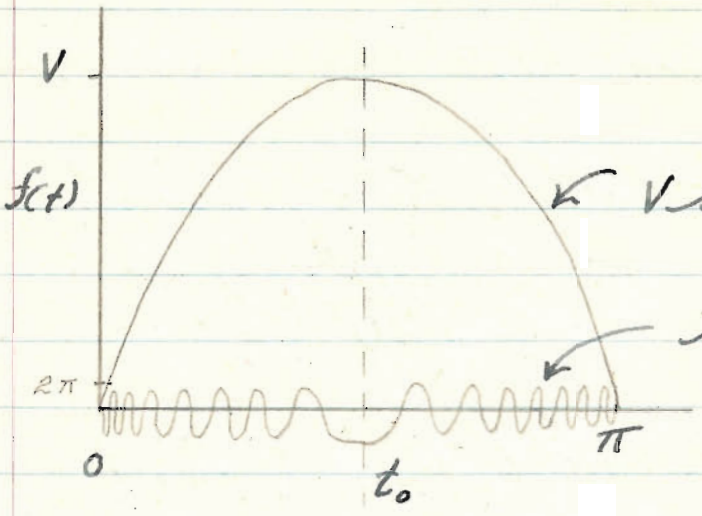
$$J_1 = -J_0'$$

From the above relations, it is possible to write $J_n(v)$ as a series:

$$J_n(v) = \sum_0^n \frac{(-1)^k}{(n+k)!\,k!}\left(\frac{v}{2}\right)^{n+2k}$$

We shall now find an approximate value of $J_n(v)$ for large v. Let us investigate graphically the behavior of integrand in the equation:

$$J_n(v) = \frac{1}{\pi}\int_0^\pi \cos(nt - v\sin t)\,dt$$



Assume $n = 0$

It can be seen that for large V the only important contribution to the integral comes from the region around t_0. Let us expand $nt - V\sin t$ around t_0.

$$f(t_0 + x) = f(t_0) + \frac{x^2}{2!} f''(t_0)$$

$$\int_0^{\pi} \cos f(t)\,dt \simeq \int_{-\infty}^{\infty} \cos\left[f(t_0) + \frac{x^2}{2} f''(t_0)\right] dx$$

Since the contributions far from t_0 will be very small we replace the limits 0 and π by $\pm\infty$.

For our case:

$$\frac{d}{dt}(nt - V\sin t) = 0$$

$$n = v\cos t_0$$

Max. at, $t_0 = \cos^{-1}\frac{n}{v}$

Let us look at the cases

$$n/v \ll 1$$

Then $t_0 \sim \frac{\pi}{2} - \frac{n}{v}$

$$(nt - v\sin t) \sim n\left(\frac{\pi}{2} - \frac{n}{v}\right) - v\sin\left(\frac{\pi}{2} - \frac{n}{v}\right) + \frac{x^2}{2}\sin\left(\frac{\pi}{2} - \frac{n}{v}\right)$$

$$\sim \frac{n\pi}{2} - v\left(1 - \frac{x^2}{2}\right)$$

We may also proceed in a different manner to get the above result:

Let

$$t = \frac{\pi}{2} - \frac{n}{v} + x$$

$$\sin\left(\frac{\pi}{2} - \frac{n}{v} + x\right) = \cos\left(\frac{n}{v} - x\right)$$

$$= 1 - \frac{1}{2}\left(\frac{n}{v} - x\right)^2 = 1 - \frac{1}{2}\frac{n^2}{v^2} + \frac{nx}{v} - \frac{x^2}{2}$$

$$(nt - v\sin t) \cong \frac{n\pi}{2} - \frac{n^2}{v} + nx - v + \frac{1}{2}\frac{n^2}{v} - nx + \frac{vx^2}{2}$$

$$\sim \frac{n\pi}{2} - v\left(1 - \frac{x^2}{2}\right)$$

Hence for large $V > n$

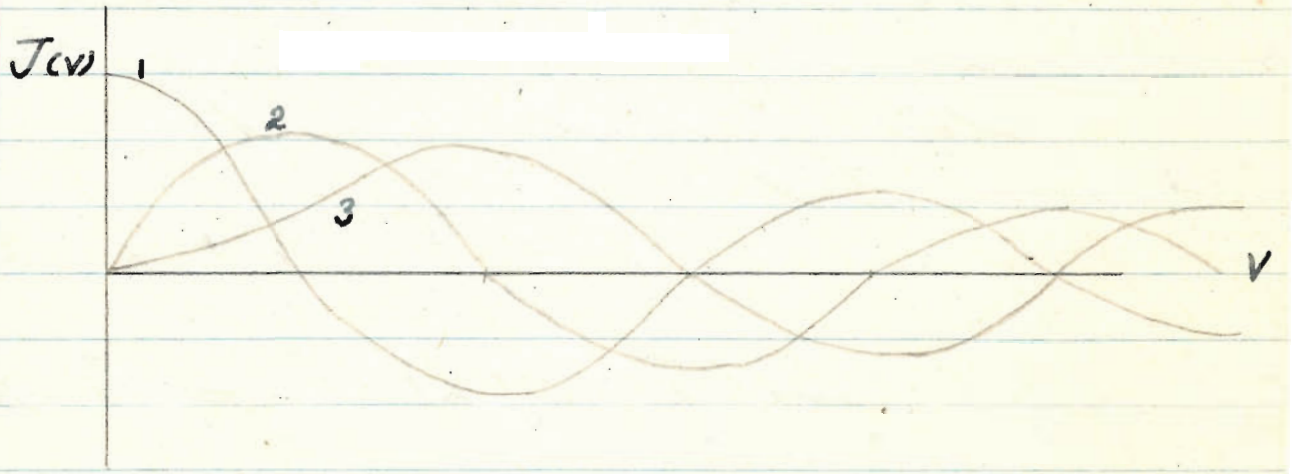
$$\pi J_n(V) \sim \int_{-\infty}^{\infty} \cos\left(\frac{n\pi}{2} - V + \frac{Vx^2}{2}\right) dx$$

$$= \int_{-\infty}^{\infty} \left[\cos\left(\frac{n\pi}{2} - V\right)\cos\frac{Vx^2}{2} - \sin\left(\frac{n\pi}{2} - V\right)\sin\frac{Vx^2}{2}\right] dx$$

$$\int_{-\infty}^{\infty} \cos\frac{Vx^2}{2}\,dx = \int_{-\infty}^{\infty} \cos\frac{Vx^2}{2}\,dx = \sqrt{\frac{\pi}{V}}$$

$$J_n(V) \sim \sqrt{\frac{1}{\pi V}}\left[\cos\left(\frac{n\pi}{2} - V\right) - \sin\left(\frac{n\pi}{2} - V\right)\right]$$

$$J_n(V) \sim \sqrt{\frac{2}{\pi V}}\cos\left(\frac{n\pi}{2} + \frac{\pi}{4} - V\right)$$



The Bessel functions look approximately as pictured above. At large V they have the same amplitude but different phase. At large V their period is two π

II Application of Bessel Functions

Bessel functions arise in problems in which there is cylindrical symmetry such as vibration of drum heads, heat flow in circular rods, and the flow of liquids in tubes.

Math Methods

Lecture XXVII Dec 13, 1946

I Vibration of a string and drum

A The wave equation

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

There are two methods available for solving this equation:

1) Find the periodic solutions - fundamental modes - and then combine them in such away that the boundary conditions are satisfied.

2) Green's method - Find solutions with a δ-function distribution at $t = 0$ and then combine them so that the boundary conditions are satisfied.

We shall make use of the first method in solving the problems which

follow.

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

The time dependence of ϕ is easily guessed

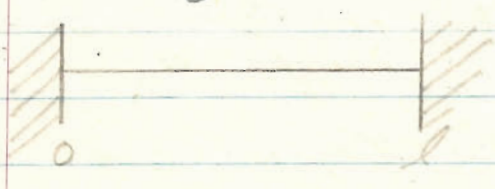
$$\phi(x,y,z,t) = f(xyz)\, e^{i\omega t}$$

Making this substitution the wave equation reduces to

$$\nabla^2 f(xyz) = -k^2 f \quad , k = \frac{\omega}{c}$$

B Vibrating String

Consider a string of length, l, fixed at both ends.



$$\frac{d^2 f}{dx^2} = -k^2 f$$

$$f = A e^{ikx}$$

$$f = A \sin kx$$

$f = 0$ at $x = 0, l$

$$kl = n\pi$$

$$\omega_n = n \frac{\pi c}{l}$$

C Vibrating Drum Head.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -k^2 f$$

Try solution $f = A e^{ik_1 x} e^{ik_2 y}$

$$k_1^2 + k_2^2 = k^2 .$$

For a square drum head of length, a, on a side the boundary conditions are $f = 0$ for $x = 0, a$ $y = 0, a$.

$$f = A \sin k_1 x \sin k_2 y$$
$$k_1 = n_1 \pi / a \qquad k_2 = n_2 \pi / a$$
$$k^2 = \frac{\pi^2}{a^2}(n_1^2 + n_2^2)$$

For a circular drum head the boundary condition is $f = 0$ $(x^2 + y^2)^{\frac{1}{2}} = a$. This can only be satisfied by a solution consisting of a sum of solutions of the type found above.

$$F = \int A(k_1) e^{i k_1 x + \sqrt{k^2 - k_1^2}\, y}\, dk_1$$

It can be shown that for a circular drum head this solution reduces to the Bessel Function.

Let us try to solve the problem of a circular drum head by a different method. Consider the two dimensional wave equation in polar coordinates:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}\right] f = -k^2 f$$

$f = R(\rho) e^{i n \theta}$ If the solution is to be unique at any point in space n must be an integer.

$$\frac{d^2 R}{d \rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{n^2}{\rho^2}\right) R = 0$$

The substitution $k\rho = v$ reduces this to the Bessel Equation.

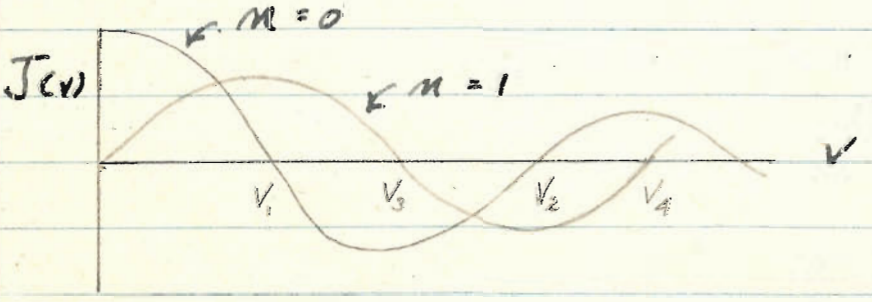
The solution for f is:

$$f = \sum [A_n J_n(kP) + B_n N_n(kP)] e^{in\theta}$$

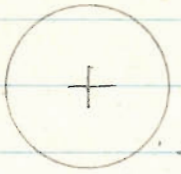
Here N_n is the Neuman Function the other solution of the Bessel Equation. The Neuman Function however goes to infinity at $P = 0$ and hence cannot be a part of the solution of our problem $\therefore B_n = 0$

$$f = \sum A_n J_n(kP) e^{in\theta}$$

Let us look at some of the solutions: normal modes. We have $f = 0$ when $P = a$



$n = 0 \qquad f = J_0(kp)$

$k = \frac{v_1}{a}$  $k = \frac{v_2}{a}$ 

$n = 1 \qquad f = J_1(kp) \begin{matrix} \sin \\ \cos \end{matrix} \theta$

$k = \frac{v_3}{a}$  $k = \frac{v_4}{a}$ 

Superposition:

$J_1 \cos\theta \qquad J_1 \sin\theta \qquad J_1(\sin\theta + \cos\theta)$

$n = 2 \qquad f = J_2(kp) \begin{matrix} \sin \\ \cos \end{matrix} 2\theta$

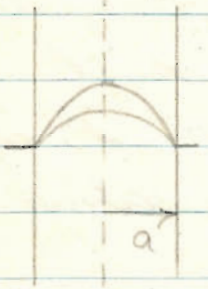


Math Methods

Lecture 28 Dec. 16, 1946

I Bessel Functions

We wish to find the temperature distribution as a function of time in a heated rod which is plunged into a quenching bath. The temperature obeys the equation



$$\nabla^2 T = \frac{\partial T}{\partial t}$$

Let us look for solutions of the form:

$$T(xyt) = f(xy)e^{-\beta t}$$

This corresponds solving the problem in terms of "normal modes" which can later be superimposed to give the appropriate solution for the given boundary conditions.

The equation for f becomes

$$\nabla^2 f(x,y) = -\beta f$$

We treated this equation in the last lecture. The solutions were found to be Bessel and Neumann functions.

A Problem: (due 1 wk after Xmas)

Consider a circular rod the surface of which is being alternately heated and cooled according to the equation

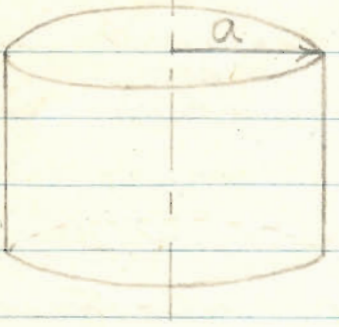
$$T_{surface} = T_0 e^{i\omega t}$$

Find the steady state solutions. Assume

$$T(x,y,t) = f(x,y) e^{i\omega t}$$

(The problem corresponds to skin effect in electro-magnetic theory)

Simplified Cavity Resonator



Consider a function $\phi(xyzt)$ which obeys the equation

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

inside a pill box and which is zero on the surface of the box. We wish to find its value inside the box.

Assume:

$$\phi = f(xyz)\,e^{i\omega t}$$

then:

$$\nabla^2 f = -k^2 f \qquad k = \frac{\omega}{c}$$

Cylindrical coordinates are appropriate for the geometry:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2} = -k^2 f$$

Separate $f = f_1(z)\,f_2(\rho\theta)$

The solution is:

$$f = a \begin{matrix}\sin\\ \cos\end{matrix} k_z z\; J_n(k_\rho \rho)\, e^{in\theta}$$

$$+ b \begin{matrix}\sin\\ \cos\end{matrix} k_z z\; N_n(k_\rho \rho)\, e^{in\theta}$$

The following conditions on ψ must hold.

ψ must be single valued and finite everywhere in the pill box.

$\psi = 0$ for $z = 0$ and b, $\rho = a$

Applying these restrictions our expression for ψ reduces to

$$\psi = a \sin k_z z \, J_n(k_\rho \rho) e^{in\theta}$$

where $k_z = m\pi/b$, $k_\rho = x_{nl}/a$, and n, m are integers. x_{nl} is the l^{th} root of the n^{th} order Bessel Function. The frequencies of the various modes are

$$k^2 = k_z^2 + k_\rho^2$$

$$\frac{\omega^2}{c^2} = \left(\frac{m\pi}{b}\right)^2 + \left(\frac{x_{nl}}{a}\right)^2$$

The lowest mode is for $n = 0$, $l = 1$, and $m = 1$.

Suppose we had had the boundary conditions

$\frac{\partial \phi}{\partial z} = 0$ for $z = 0$ and b

$\frac{\partial \phi}{\partial \rho} = 0$ for $\rho = a$

With these boundary conditions our solution for f becomes

$$f = a \cos k_z z \, J_n(k_\rho \rho) \, e^{in\theta}$$

where $k_z = \frac{m\pi}{b}$, $k_\rho = \frac{x'_{nl}}{a}$

x'_{nl} are the roots of the equation:

$$J_n'(x) = 0$$

Problem: (due one wk after Xmas) Simplified Wave Guide Problem.

An infinitely long hollow cylinder carries waves down it which satisfy the equation $\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$. These waves have frequency ω. What is their velocity down the pipe if $\phi = 0$ at $r = a$. Assume $\phi = f(\rho\theta) \cos \omega (t - \frac{z}{v})$. Note the velocity v varies from mode to mode. How would interpret the imaginary

velocities arising if a is too small? If you desire only to transmit the fundamental mode, within what limits of size must the radius, a, be.

II Bessel Function defined for non integral n

$$J_n(z) = \frac{2}{\sqrt{\pi}} \frac{\left(\frac{z}{2}\right)^n}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^{\pi/2} \cos(z\cos\phi) \sin^{2n}\phi \, d\phi$$

Demonstrate recursion relations for this definition.

III Complete Solution of Bessel Equation.

$$y = A J_n(x) + B J_{-n}(x)$$

If n is not an integer. When n is an integer we have:

$$J_{-n}(x) = (-1)^n J_n(x)$$

Math Methods

Lecture 29 Dec 18

I Bessel Function

$$J_n(v) = \frac{2}{\sqrt{\pi}} \frac{(v/2)^n}{\Gamma(n+\frac{1}{2})} \int_0^{\pi/2} \cos(v\cos\phi) \sin^{2n}\phi \, d\phi$$

The complete solution of the Bessel Equation for non-integral n is:

$$A J_n + B J_{-n}$$

The series for the Bessel Function for non-integral n is obtained by replacing the $n!$ in the series for integral by $\Gamma(n+1)$:

$$J_n(v) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{v}{2}\right)^{n+2k}}{\Gamma(k+1)\Gamma(n+k+1)}$$

For integral n we have:

$$J_n = (-1)^n J_{-n}$$

J_n and J_{-n} are not linearly

independent for integral n. The Neumann Function is defined for integral n by:

$$N_n(x)\sin n\pi = J_n(x)\cos n\pi - J_{-n}(x)$$

For integral n the complete solution of the Bessel Equation is:

$$\alpha J_n + \beta N_n$$

N_n obeys the same recursion relations as J_n.

$$N_0(x) = \left(\ln(x) J_0(x) + \frac{x^2}{2^2} - \frac{x^4(1+\frac{1}{2})}{2^2 4^2} + \frac{x^6(1+\frac{1}{2}+\frac{1}{3})}{2^2 4^2 6^2} - \ldots\right)$$

$N_0(x)$ has a $\ln$ singularity at $x=0$. $N_n(x)$ behaves as A/x^n near $x=0$.

$$N_{p-1} J_p - N_p J_{p-1} = 2/\pi x$$

For large x

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(\frac{\pi n}{2} + \frac{\pi}{4} - x\right)$$

$$N_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(\frac{\pi n}{2} + \frac{\pi}{4} - x\right)$$

Integral Definitions

$$N_0(x) = -\frac{2}{\pi} \int_0^\infty \cos(x \cosh u)\, du$$

$$J_0(x) = \frac{2}{\pi} \int_0^\infty \sin(x \cosh u)\, du$$

The following functions are not Bessel Functions. They are tabulated.

Lommel Weber

$$U_p(x) = \frac{1}{\pi} \int_0^\pi \sin(x \sin\phi - p\phi)\, d\phi$$

Struve

$$S_p(x) = \frac{2}{\sqrt{\pi}} \frac{(x/2)^p}{\Gamma(p+\frac{1}{2})} \int_0^{\frac{\pi}{2}} \sin(x\cos\phi) \sin^{2p}\phi\, d\phi$$

Problems involving Bessel Functions

Hanging of a chain

$$\frac{\partial}{\partial x}\left(T \frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}$$

Tension, $T = k x$

$$k \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}$$ involves Bessel Functions.

$$\frac{d^2\psi}{dx^2} + x\psi = 0$$ involves $J_{\frac{1}{3}}$

$$\frac{d^2\psi}{dx^2} + x^n \psi = 0$$ involves Bessel Functions (Jahnke Emde)

Half Integral Bessel Functions:

$$J_{\frac{1}{2}}(z) = \frac{2}{\sqrt{\pi}} \sqrt{\frac{z}{2}} \int_0^{\pi/2} \cos(z\cos\phi) \sin\phi \, d\phi$$

$$= -\frac{2}{\sqrt{\pi}} \sqrt{\frac{z}{2}} \frac{1}{z} \sin(z\cos\phi)\Big|_0^{\pi/2}$$

$$= \sqrt{\frac{2}{\pi z}} \sin z$$

All $\frac{1}{2}$ integral Bessel Functions may be expressed

in terms of elementary functions

$$H_n^{(1)}(x) = J_n(x) + i\,N_n(x)$$

$$H_n^{(2)}(x) = J_n(x) - i\,N_n(x)$$

ber and bei

$$J_0(x\sqrt{-i}) = \mathrm{ber}\,x + i\,\mathrm{bei}\,x$$

Continued Fractions

$$x = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1\cdots}}}}$$

$$x = 1 + \frac{1}{x}$$

$$x = (1+\sqrt{1+4})/2 = 1.586$$

$$1 + \cfrac{1}{2x+\cfrac{1}{3x+\cfrac{1}{4x+\cfrac{1}{5x\cdots}}}} = \frac{J_1(-\frac{2i}{x})}{i\,J_0(-\frac{2i}{x})}$$

Math Methods

Lecture 30 Dec 20, 1946

I Solution of Linear Equations

A. $$n P_n(V) = \frac{V}{2}\left[P_{n+1}(V) + P_{n-1}(V)\right]$$

We define the Generating Function, f, as follows:

$$f(Vt) = \sum_{-\infty}^{\infty} P_n t^n$$

(Sometimes an $n!$ is useful in defining Generating Functions)

$$\frac{\partial f}{\partial t} = \sum n t^{n-1} P_n$$

$$\sum n t^n P_n(V) = \frac{V}{2}\sum\left[t^n P_{n+1}(V) + t^n P_{n-1}(V)\right]$$

$$t\frac{\partial f}{\partial t} = \frac{V}{2}\left[\frac{1}{t} + t\right] f$$

$$\frac{\partial f}{f} = \frac{V}{2}\left[\frac{1}{t^2} + 1\right] dt$$

$$f = C(V)\, e^{\frac{V}{2}\left(t - \frac{1}{t}\right)}$$

$$\sum_{-\infty}^{\infty} \frac{1}{C(V)} P_n(V) t^n = e^{\frac{V}{2}\left(t - \frac{1}{t}\right)}$$

$$J_n(v) = \frac{P_n(v)}{C(v)}$$

$$\sum_{-\infty}^{\infty} t^n J_n(v) = e^{\frac{v}{2}\left(t - \frac{1}{t}\right)}$$

B $\quad \frac{d}{dv} Q_n(v) = -\frac{1}{2}\left[Q_{n+1}(v) - Q_{n-1}(v)\right]$

$$f(vt) = \sum Q_n(v)\, t^n$$

Suppose $\quad Q_n(0) = 0, \quad Q_0(0) = 1$

$$f(0t) = 1$$

$$\sum t^n Q_n' = -\frac{1}{2}\sum\left[t^n Q_{n+1} - t^n Q_{n-1}\right]$$

$$\frac{\partial f}{\partial v} = -\frac{1}{2}\left(\frac{1}{t} - t\right) f$$

$$\frac{\partial f}{\partial v} = \frac{1}{2}\left(t - \frac{1}{t}\right) f$$

$$f(vt) = S(t)\, e^{\frac{v}{2}\left(t - \frac{1}{t}\right)}$$

$$f(0t) = 1 = S(t)$$

$$e^{\frac{v}{2}\left(t - \frac{1}{t}\right)} = \sum Q_n(v)\, t^n$$

$$Q_n(v) \equiv J_n(v)$$

Consider:

$$J_n(Y+Z) = \sum_{m=-\infty}^{\infty} J_m(Y) J_{n-m}(Z)$$

Proof: $\sum t^n J_n(Y+Z) = e^{\frac{Y+Z}{2}(t-\frac{1}{t})}$

$$= e^{\frac{Y}{2}(t-\frac{1}{t})} e^{\frac{Z}{2}(t-\frac{1}{t})}$$

$$= \sum_k t^k J_k(Y) \sum_l t^l J_l(Z)$$

Equating like powers of t

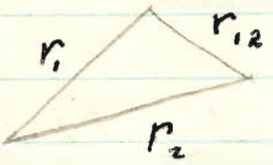
$$J_n(Y+Z) = \sum_k J_k(Y) J_{n-k}(Z)$$

II Legender Polynomials

$$\sum_0^{\infty} P_n(x) t^n = (1 - 2tx + t^2)^{-\frac{1}{2}}$$

Using the Binomial Expansion expand in a series in t. The coefficient of t^n is the n^{th} order Legender Polynomial.

$$1 + tx + t^2\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right) + \ldots$$
$$P_0(x) \quad P_1(x) \quad P_2(x)$$



$$\frac{1}{|\vec{r}_{12}|} = \sum_0^\infty \frac{|r_1|^K}{|\vec{r}_2|^{K+1}} P_K(\cos\theta)$$

$$|r_1| < |r_2|$$

Math Methods

Lecture 31 Jan 6 1997

I Legendre Polynomials

$$1/\sqrt{1-2tx+t^2} = \sum_0^\infty t^n P_n(x)$$

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

$$f(xt) = \sum_0^\infty t^n P_n(x)$$

$$\sum (n+1)t^n P_{n+1}(x) - \sum (2n+1)t^n x P_n(x)$$

$$+ \sum n t^n P_{n-1}(x) = 0$$

$$\frac{\partial f}{\partial t} = \sum_0^\infty (n+1)t^n P_{n+1}$$

$$2xt\frac{\partial f}{\partial t} = \sum_0^\infty 2nxt^n P_n$$

$$xf = \sum_0^\infty xt^n P_n$$

$$t\frac{\partial (ft)}{\partial t} = \sum_0^\infty n t^n P_{n-1}$$

$$(1-2xt+t^2)\frac{\partial f}{\partial t} = (x-t)f$$

$$f(xt) = C(x) / \sqrt{1 - 2xt + t^2}$$

Suppose $P_0(x) = 1 = f(x0)$.
Hence $C(x) = 1$

Home Prob. Show:

$$(1 - x^2) P_n'(x) + nx P_n - n P_{n-1} = 0$$

Consider:

$$\int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta$$

$$= \int_{-1}^{+1} P_n(x) P_m(x) \, dx$$

$$= \frac{2}{2n+1} \delta_{mn}$$

Proof: Let $I_{nm} = \int_{-1}^{+1} P_n P_m \, dx$
Consider:

$$\sum_{nm} t^n s^m I_{nm}$$

$$= \sum_{nm} \int_{-1}^{+1} t^n s^m P_n P_m \, dx$$

$$= \int_{-1}^{+1} dx / \sqrt{1 - 2xt + t^2} \cdot \sqrt{1 - 2xs + s^2}$$

$$= \frac{2}{\sqrt{4st}} \ln \left[\sqrt{4st(1-2xt+t^2)} - 2t\sqrt{1-2xs+s^2} \right] \Big|_{-1}^{+1}$$

$$\sum_{n\,m} t^n s^m I_{nm} = \frac{1}{\sqrt{st}} \ln\left(\frac{1+\sqrt{st}}{1-\sqrt{st}}\right)$$

Expand in powers of st

$$\ln\left(\frac{1+x}{1-x}\right) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \right] \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right]$$

$$\sum_{n\,m} t^n s^m I_{nm} = 2 + \frac{2st}{3} + \frac{2s^2t^2}{5} + \cdots$$

$$I_{nm} = \frac{2}{(2n+1)} \delta_{nm} \qquad \text{Q.E.D.}$$

Home Problem. Show:

$$\int_{-1}^{+1} P_n(x) P_m(x) x\, dx$$

$$= 0 \text{ unless } n = \begin{pmatrix} m+1 \\ m-1 \end{pmatrix}$$

$$= \left(\frac{2}{2m+1}\right) \cdot \frac{m+1}{2m+3}$$

if $n = m+1$

$$= \left(\frac{2}{2m+1}\right) \cdot \frac{m}{2m-1}$$

if $n = m-1$

A Differential Equation involving Legendre Polynomials

$$\frac{d}{dx}\left\{(1-x^2)\frac{dz}{dx}\right\} + n(n+1)z = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP_n(\cos\theta)}{d\theta}\right).$$

$$+ n(n+1)P_n(\cos\theta) = 0$$

Spherical Cavity Resonator

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

The Laplacian in polar coordinates

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

The term $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)$ may be written

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)$$

$$\phi = f(r\,\theta\,\varphi)\,e^{i\lambda t}$$

$$\nabla^2 f = -k^2 f$$

Suppose f independent of ϕ.
Try solution

$$f = g(r) P_n(\cos\theta)$$

$$P_n(\cos\theta) \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dg}{dr}\right) + \frac{g}{r^2} \frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{\partial P_n}{\partial \theta}\right)$$

$$= -k^2 g P_n(\cos\theta)$$

The equation separates into a function of r and a function of θ

$$0 = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial P_n(\cos\theta)}{\partial\theta}\right) + k_\theta^2 P_n(\cos\theta)$$

$$k_\theta^2 = n(n-1)$$

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dg}{dr}\right) - \left(\frac{k_\theta^2}{r^2} - k^2\right) g = 0$$

Show that the solution of the r equation is

$$g(r) = \sqrt{\frac{\pi}{2kr}} \left[A J_{n+\frac{1}{2}}(kr) + B N_{n+\frac{1}{2}}(kr) \right]$$

Math Methods

Lecture 32 Jan 8, 1947

I Associated Legendre Functions

The solutions of the differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P_n^{|m|}}{\partial\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]P_n^{|m|}$$

are the associated Legendre Functions $P_n^{|m|}(\cos\theta)$

$$P_n^{|m|}(\cos\theta)\equiv\sin^{|m|}\theta\left[\frac{d^{|m|}}{dx^{|m|}}P_n(x)\right]_{x=\cos\theta}$$

II Spherical Cavity Resonator

$$\nabla^2 f=-k^2 f$$

This equation separates in polar coordinates. The solution for f is

$$f=\sqrt{\frac{\pi}{2kr}}\left[AJ_{n+\frac{1}{2}}(kr)+BN_{n+\frac{1}{2}}(kr)\right]P_n^{|m|}(\cos\theta)\,e^{im\phi}$$

f is finite at the origin hence $B=0$

Also $f = 0$ at $r = a$

$$J_{n+\frac{1}{2}}(ka) = 0$$

$$k = X^{l}_{n+\frac{1}{2}} / a = \omega$$

$$m \leq n$$

Case I $n = 0$ $m = 0$

$$Y_{00}(\theta\phi) = [P^{0}_{0}(\cos\theta) = 1][e^{i0} = 1] = 1$$

Case II $n = 1$ $m = 0$

$$Y_{10}(\theta\phi) = \cos\theta$$

$$n = 1 \quad m = \pm 1$$

$$Y_{11}(\theta\phi) = \sin\theta \begin{Bmatrix} \cos\phi \\ \sin\phi \end{Bmatrix}$$

Case III $n = 2$ $m = 0, \pm 1 \pm 2$

$$P_2^0 = \frac{3}{2}\cos^2\theta - \frac{1}{2}$$

$$P_2^1 = 3\sin\theta\cos\theta$$

$$P_2^2 = 3\sin^2\theta$$

$$Y_{20}(\theta\phi) = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_{21}(\theta\phi) = 3\cos\theta\sin\theta\, e^{\pm i\phi}$$

$$Y_{22}(\theta\phi) = 3\sin^2\theta\, e^{\pm 2i\phi}$$

All solutions may be obtained by a linear combination of the type of solutions found above. This follows because if we make a coordinate transformation $\theta \to \theta'$ $\phi \to \phi'$

$$Y_n'(\theta'\phi') = \sum_{m'} \gamma_{nm'} Y_{nm'}(\theta\phi)$$

Home Problem: Find the frequencies of the above modes.

III Useful Relations about Legendre Functions

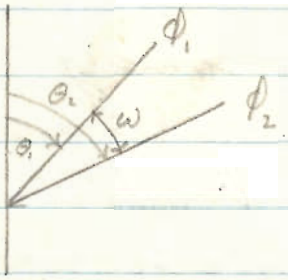
A. $$\frac{1}{r^{n+1}} P_n(\cos\theta) = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \cos\theta = \frac{z}{r}$$

B. $$P_n(x) = \frac{1}{2^n n!} \frac{d}{dx^n}\left[(x^2-1)^n\right]$$

C. $$P_n(\cos\omega)$$

$$= \sum_{m=-n}^{+n} \frac{(n-|m|)!}{(n+|m|)!} P_n^{|m|}(\cos\theta_1) P_n^{|m|}(\cos\theta_2) e^{im(\phi_1-\phi_2)}$$



$$\cos\omega = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$$

The second solution of the Legendre Equation is

$$Q_n(z) = \frac{1}{2}\int_{-1}^{+1} \frac{P_n(y)}{z-y}\,dy$$

Math Methods

Lecture 33 Jan. 10, 1947

I Fourier Series

Suppose we have found special solutions of the heat equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

of the form $T = a \begin{matrix}\sin \\ \cos\end{matrix} kx \, e^{-k^2 t}$

Imposing the boundary conditions $T = 0 : x = 0, a$ and $T(x\,0) = f(x)$ we have as a solution of our problem

$$T(x\,t) = \sum_0^\infty a_n \sin \frac{n\pi x}{a} \, e^{-\frac{\pi^2 n^2}{a^2} t}$$

where $f(x) = T(x\,0) = \sum_0^\infty a_n \sin \frac{n\pi x}{a}$

We may evaluate the a_n above as follows:

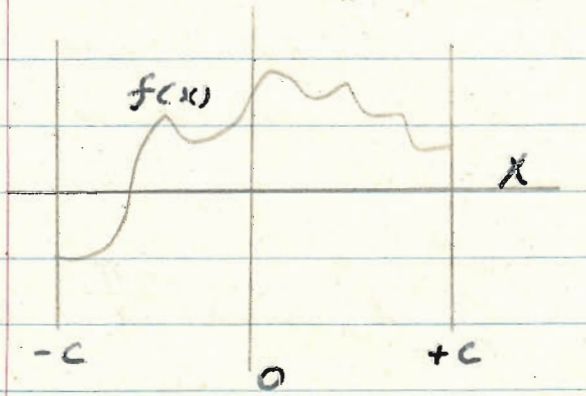
$$\int_0^a \sin \frac{m\pi x}{a} f(x)\,dx = \sum_0^\infty a_n \int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a}\,dx$$

but $\int_0^a \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{a}\,dx = \frac{1}{2}a\,\delta_{nm}$

hence: $a_m = \frac{2}{a}\int_0^a \sin\frac{m\pi x}{a} f(x)\,dx$

series for $f(x)$ is known as a Fourier Series. The a_m are the Fourier coefficients.

Let us consider a general function not necessarily zero at the limits.



This function may be expanded in terms of both sines and cosines.

$$f(x) = \frac{1}{2}b_0 + \sum_n \left[a_n \sin\frac{n\pi x}{c} + b_n \cos\frac{n\pi x}{c}\right]$$

$$b_m = \frac{1}{c}\int_{-c}^{+c} f(x)\cos\frac{m\pi x}{c}\,dx$$

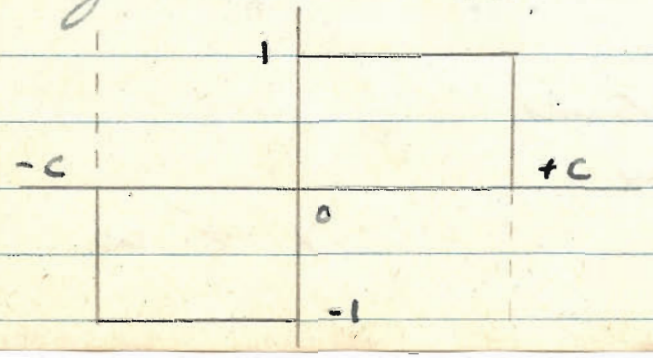
$$a_n = \frac{1}{c}\int_{-c}^{+c} f(x)\sin\frac{n\pi x}{c}\,dx$$

Any function which is single valued and has only a finite number of discontinuities may be expanded in a Fourier Series in a given interval, 2c. The Fourier Series gives a function which is periodic in 2c.

Symmetric functions will involve only cosines. Antisymmetric functions will involve only sines.

Ref: Sokolinkoff. Quick easy methods for numerical computation of Fourier coefficients.

Fourier Expansion of a Square Wave.



$f(x) = -1 \quad -c < x < 0$

$f(x) = 1 \quad 0 < x < c$

Since $f(x)$ is asymmetric only sines will be involved

$$a_m = \frac{1}{c}\int_{-c}^{+c} f(x)\sin\frac{\pi m x}{c}\,dx$$

$$= -\frac{1}{c}\int_{-c}^{0}\sin\frac{m\pi x}{c}\,dx + \frac{1}{c}\int_{0}^{c}\sin\frac{m\pi x}{c}\,dx$$

$$= -\frac{1}{c}\frac{c}{m\pi}\left(-\cos\frac{m\pi x}{c}\right)_{-c}^{0} + \frac{1}{c}\frac{c}{m\pi}\left(-\cos\frac{m\pi x}{c}\right)_{0}^{c}$$

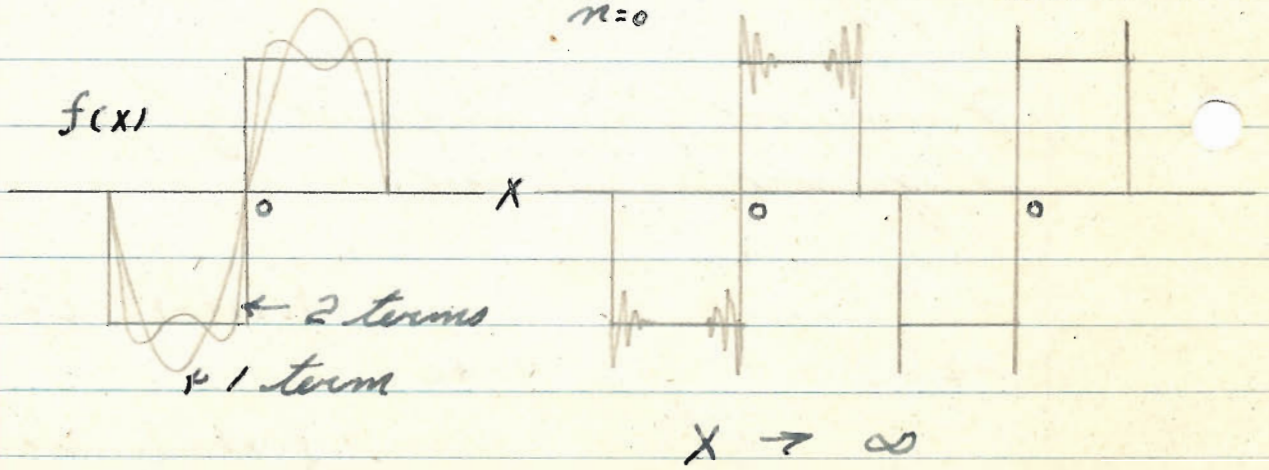
$$= \frac{1}{m\pi}\left(1-[-1]^m\right) + \frac{1}{m\pi}\left(1-[-1]^m\right)$$

$$a_m = \frac{2}{m\pi}\left(1-[-1]^m\right)$$

If m even $\quad a_m = 0$.

If m odd $\quad a_m = \frac{4}{m\pi}$

$$f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\left[\sin\frac{(2n+1)\pi x}{c}\right]\frac{1}{(2n+1)}$$



The oscillations near 0 have nothing to do with physics. At any finite distance, ϵ, from 0 they can be made as small as one wishes. In any physical problem there would be a direct effect corresponding to Abel summation which would smooth out the wiggles.

Math Methods

Lecture 34 Jan 13, 1947

I Fourier Series

We saw last lecture that any single valued real function having a countable set of discontinuities could be expanded in a series involving sins and cosins, a Fourier Series.

We wish now to consider the Fourier Expansion of a function in a complex series:

$$f(x) = \sum_{-\infty}^{\infty} a_n e^{\frac{i n \pi x}{c}}$$

$$-c \leq x \leq c$$

$$a_n = \frac{1}{2c} \int_{-c}^{+c} f(x) e^{-\frac{i n \pi x}{c}} dx$$

Some properties of Fourier Series

$$f(x) = \sum_{-\infty}^{\infty} \frac{1}{2c} e^{i\frac{n\pi x}{c}} \int_{-c}^{c} f(y) e^{-\frac{i n\pi y}{c}} dy$$

$$f(x) = \int_{-c}^{+c} f(y) \sum_{-\infty}^{\infty} \frac{1}{2c} e^{\frac{i n\pi}{c}(x-y)} dy$$

Hence $\delta(x-y) = \sum_{-\infty}^{\infty} \frac{1}{2c} e^{\frac{i n\pi}{c}(x-y)}$.

Suppose we have represented a function E in the form

$$E = \sum_{-\infty}^{\infty} a_n e^{\frac{i n\pi x}{c}}$$

If E represents the voltage across a resistor, the power dissipated in the resistor will be proportional to the real part of E squared or EE^* averaged over a period.

$$W = \frac{1}{2\pi} \int_{-\pi}^{\pi} E E^* \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{-\infty}^{\infty} a_n e^{inx} \sum_{-\infty}^{\infty} a_m^* e^{-imx} \, dx$$

$$= \sum_{nm} a_n a_m^* \delta_{nm}$$

$$W = \sum_n |a_n|^2$$

Problems:

I. Show:

$$\sum_{-N}^{N} a_n e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin\frac{1}{2}(x-t)} f(t)\,dt$$

II A half wave rectifier is connected in series with a resistor. A 60~ sine wave is impressed on the combination. Find: average D.C. power; and the power in the 60, 120, 180, 240~ components.

Repeat for a full wave rectifier.

III Using the Fourier Expansion for a Square Wave sum the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

IV Show that if

$$f(x) = \sum a_n e^{inx}$$

$$g(x) = \sum b_m e^{imx}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,g^*(x)\,dx = \sum_n a_n b_n^*$$

Differentiation of a Fourier Series:

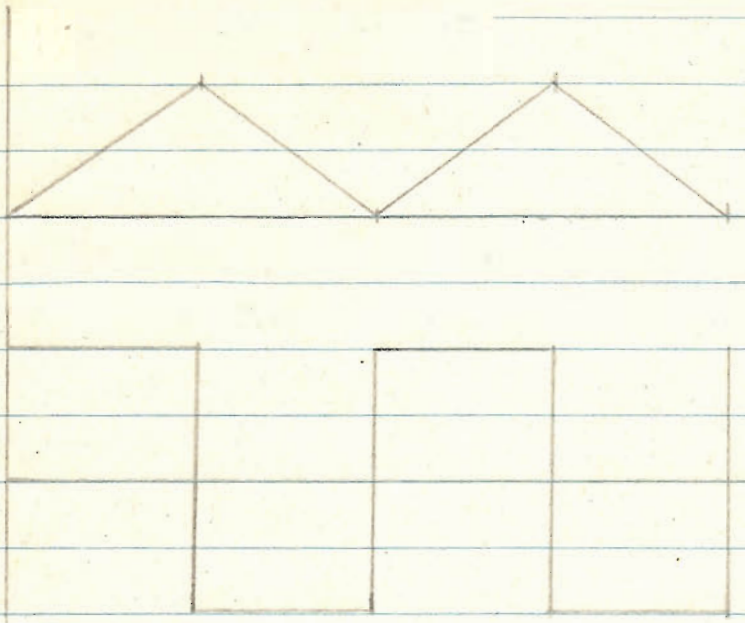
$$f(x) = \sum a_n e^{inx}$$

$$f'(x) = \sum in\,a_n e^{inx}$$

Integration of a Fourier Series:

$$\int^x f(x)\,dx = \sum_{n\neq 0} \frac{a_n}{in} e^{inx} + a_0 x$$

Illustrative problem involving use of integration of a Fourier Series.



$$f'(x) = \frac{4A}{\pi}\sum_{0}^{\infty} \frac{1}{(2n+1)} \sin \frac{(2n+1)\pi x}{c}$$

$$c + f(x) = -\frac{4Ac}{\pi^2} \sum_{0}^{\infty} \frac{1}{(2n+1)^2} \cos \frac{(2n+1)\pi x}{c}$$

$$f(x) = \frac{4Ac}{\pi^2} \sum_{0}^{\infty} \frac{1}{(2n+1)^2} \left[1 - \cos \frac{(2n+1)\pi x}{c}\right]$$

Math Methods

Lecture 35 Jan 15 1947

Fourier Transforms

Suppose we wish to represent a function on an infinite interval. We know how to represent it in terms of a Fourier Series on the finite interval $-c$ to $+c$:

$$f(x)\Big|_{-c}^{c} = \sum_{-\infty}^{\infty} a_n e^{\frac{i n \pi x}{c}}$$

$$a_n = \frac{1}{2c}\int_{-c}^{c} f(x) e^{-\frac{i n \pi x}{c}} dx$$

Let us attempt to find the limit of this series as $c \to \infty$.

Let $\omega = \frac{n\pi}{c}$ Then for large c

$$\sum_{-\infty}^{\infty} a_n e^{\frac{i n \pi x}{c}} \to \int_{-\infty}^{\infty} a_\omega \frac{c}{\pi} e^{i\omega x} d\omega$$

$$a_\omega \frac{c}{\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$\frac{c}{\pi} a_\omega = \phi(\omega)$$

Hence we have for x in the range $-\infty$ to ∞

$$f(x) = \int_{-\infty}^{\infty} \phi(\omega) e^{i\omega x} d\omega \quad \left[\frac{1}{\sqrt{2\pi}}\right]$$

$$\phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \quad \left[\frac{1}{\sqrt{2\pi}}\right]$$

If we substitute the second of these equations in the first it should lead to an identity. The demonstration here given is not rigorous (it contains two mistakes which cancel) but leads to the correct result.

$$f(x) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-i\omega y} dy \right\} e^{i\omega x} d\omega$$

$$\stackrel{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y) e^{i\omega(x-y)} dy\, d\omega$$

$$\stackrel{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(x-y) f(y)\, dy$$

$$f(x) \equiv f(x)$$

Prove the following:

If $f(x) = T\phi(\omega)$

$$\int_{-\infty}^{\infty} f(x) f^*(x) dx = \int_{-\infty}^{\infty} \phi(\omega) \phi^*(\omega) d\omega$$

If $f(x) = T\phi(\omega)$ $g(x) = T\psi(\omega)$

$$\int_{-\infty}^{\infty} f(x) g^*(x) = \int \phi(\omega) \psi(\omega) d\omega$$

$$f(x) + g(x) = T(\phi(\omega) + \psi(\omega))$$

Relations between a function and its transform

Function	Transform
$f(x)$	$\phi(\omega)$
$f'(x)$	$i\omega\phi(\omega)$
$\int^x f(x) dx$	$\frac{1}{i\omega}\phi(\omega)$
$xf(x)$	$i\phi'(\omega)$
$f(x)g(x)$	$\int_{-\infty} \phi(\omega-u)\psi(u) du$
$1(x)$	$\delta(\omega)$
x	$i\delta'(\omega)$
$f''(x) + a^2 f(x)$	$(-\omega^2 + a^2)\phi(\omega)$

Any function may be Fourier Transformed which obeys the following conditions
(1) Single valued
(2) Only a countable set of discontinuities
(3) $\int_{-\infty}^{\infty} f(x) f^*(x) dx$ must be finite.

Fourier Transforms are very useful for the solution of Linear Differential Equations.

$$f'(x) + a f(x) = g(x)$$

$Tf = \phi$; $Tg = \psi$; $g(x)$ is assumed known. The transformed equation is:

$$i\omega \phi(\omega) + a \phi(\omega) = \psi(\omega)$$

$$\phi(\omega) = \psi(\omega) / (a + i\omega)$$

$$f(x) = \int_{-\infty}^{\infty} \frac{\psi(\omega)}{a + i\omega} e^{i\omega x} d\omega$$

We shall now solve an equation in detail.

$$y'' + 2y' + y = \begin{cases} 0 & x < 0 \\ e^{-2x} & x > 0 \end{cases} = g(x)$$

$$Ty = \phi \; ; \; Tg = \psi$$

The transformed equation is:

$$-\omega^2 \phi + 2i\omega\phi + \phi = \psi$$

$$\phi = \psi / (-\omega^2 + 2i\omega + 1)$$

$$\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x)\, e^{i\omega x}\, dx$$

$$\psi = 0 + \frac{1}{2\pi} \int_{0}^{\infty} e^{-(2+i\omega)x}\, dx$$

$$= \frac{1}{2\pi} \left(\frac{1}{2 + i\omega} \right)$$

$$\phi = \frac{1}{2\pi} \left(\frac{1}{2 + i\omega} \right) \left(\frac{1}{-\omega^2 + 2i\omega + 1} \right)$$

$$y = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega x} / (2 + i\omega)(-\omega^2 + 2i\omega + 1)$$

This integral may be evaluated by contour integration.

Math Methods

Lecture 36 Jan 17 1947

I Fourier Coefficients

If $f(x)$ is a real function then the following relation between the coefficients a_n in the expansion

$$f(x) = \sum_{-\infty}^{\infty} a_n e^{\frac{in\pi x}{c}}$$

is easily proved:

$$a_{-n} = a_n^*$$

II Solution of Linear Differential Equations by the method of Fourier Transforms.

$$\left[a_n D^n + a_{n-1} D^{n-1} + \dots a_1 D^1 + a_0\right] y = g(x)$$

Transforming this equation:

$$P(D)\,y = g(x)$$

$$P(i\omega)\,\Phi = \Psi$$

where $Ty = \Phi$; $Tg = \Psi$

$$\Phi(\omega) = \Psi(\omega) / P(i\omega)$$

$$y(x) = \int_{-\infty}^{\infty} \frac{\Psi(\omega)}{P(i\omega)} e^{i\omega x}\, dx\, d\omega$$

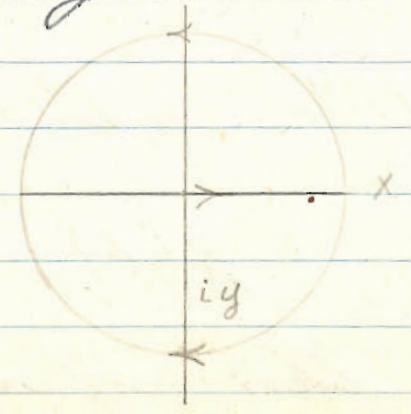
$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x') \frac{e^{i\omega(x-x')}}{P(i\omega)}\, dx'\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x')\, dx' \int_{-\infty}^{\infty} \frac{e^{i\omega(x-x')}}{P(i\omega)}\, d\omega$$

Let us attempt to evaluate the integral

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{i\omega z}}{P(i\omega)}\, d\omega$$

by contour integration.



$$F(z) = 2\pi i \sum R$$

providing we chose the appropriate contour.

The integrand under consideration will have a pole where ever $i\omega$ equals one of the roots of the polynomial $P(i\omega)$. Factoring the polynomial

$$P(i\omega) = \sum_0^N a_n (i\omega)^n$$

$$= a_N (i\omega - \alpha_1)(i\omega - \alpha_2) \ldots (i\omega - \alpha_N)$$

If the roots are all distinct there will be N simple poles esp to $\omega_k = \alpha_k / i$. The residue at the poles will be:

$$R_k = \frac{e^{\alpha_k z}}{i P'(\alpha_k)}$$

In circuit analysis, if all the resistances are positive, all the poles will be in the upper half plane.

For $z < 0$ the integral vanishes on the lower contour and since no

poles are inclosed $F(z) = 0$; $z < 0$. For $z > 0$ the integral vanishes on the upper contour and therfore $F(z) = 2\pi i \sum_K e^{\alpha_K z} / i P'(\alpha_K)$. Therfore

$$y(x) = \sum_{K=1}^{N} \int^{x} g(x') \frac{e^{\alpha_K (x - x')}}{P'(\alpha_K)} dx' + \sum_{K=1}^{N} A_K e^{\alpha_K x}$$

or:

$$y(x) = \sum_{K=1}^{N} e^{\alpha_K x} \int^{x} \frac{e^{-\alpha_K x'}}{P'(\alpha_K) g(x')} dx' + \sum_{K=1}^{N} A_K e^{\alpha_K x}$$

The above is the general solution of the differential equation:

$$P(D) y = g(x)$$

where the roots of the polynomial $P(D)$ are distinct.

Math Methods

Lecture 37 Jan 20 1947

I The solution of the differential equation

$$P(D)\,y = g(x)$$

is: $$y = \sum \int_{x'}^{x} g(x') \frac{e^{\alpha_i (x-x')}}{P'(\alpha_i)}\, dx + \sum A_i e^{\alpha_i x}$$

This equation is valid if the roots of the polynomial $P(D)$ are distinct. We can investigate what happens when two roots are identical as follows: Assume first that the roots are not quite identical

$$\alpha_1 = \alpha - \epsilon$$
$$\alpha_2 = \alpha + \epsilon$$

$$P(\alpha_1) = P(\alpha_2) = 0 \qquad P'(\alpha) = 0$$

$$P'(\alpha_1) = P'(\alpha) - \epsilon P''(\alpha)$$

$$\epsilon P''(\alpha) = P'(\alpha_1)$$
$$\epsilon P''(\alpha) = P'(\alpha_2)$$

$$y = \int^{x} g(x') \frac{e^{\alpha(x-x')}}{\epsilon P''(\alpha)} \left(e^{\epsilon(x-x')} - e^{-\epsilon(x-x')} \right) dx' + \ldots$$

$$y = \int^{x} g(x') e^{\alpha(x-x')} \frac{\epsilon(x-x') + \epsilon(x-x')}{\epsilon P''(\alpha)} dx' + \ldots$$

$$y = \int^{x} 2(x-x') g(x') \frac{e^{\alpha(x-x')}}{P''(\alpha)} dx'$$

$$+ (A + Bx) e^{\alpha x}$$

The above equation holds when there are 2 roots equal to α.

Illustrative Example

$$y'' + 2y' + y = 2(x) e^{-2x}$$

$$P(D) = D^2 + 2D + 1 = (D+1)^2$$
$$P'(D) = 2D + 2$$
$$P''(D) = 2$$

$$y = 0 \quad : \quad x < 0$$

$$y = \int_0^x 2(x-x')\, 1(x')\, e^{-2x'}\, \frac{e}{2}^{-(x-x')}\, dx'$$

$$+ (A + Bx)e^{-x}$$

$x' < x < 0 \quad y = (A + Bx)e^{-x}$

$y = 0$

$A = B = 0$

$0 < x' < x \quad y = e^{-x} \int_0^x (x-x')\, e^{-x'}\, dx'$

$$y = (x-1)e^{-x} + e^{-2x}$$

Fourier Transforms are often useful for solving differential equations in which the coefficients are not constants:

$$y'' + xy = 0$$

$$\mathcal{F} = f$$

$$i\omega \mathcal{F} = f'$$

$$i\mathcal{F}' = xf$$

$$-\mathcal{F}' = x^2 f$$

$$-\omega^2 \mathcal{F} + i\mathcal{F}' = 0$$

$$d = e^{\frac{i\omega^3}{3}}$$

$$y = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} e^{-\frac{i\omega^3}{3}} d\omega$$

This integral is a $\frac{1}{3}$ order Bessel Function. In general, the equation

$$y'' + x^n y = 0$$

has $\frac{1}{3}$ order Bessel Functions as solutions. The equation is transformed by the substitution $y = x^k Z(x)$

:

Use of Fourier Transforms in Circuit Analysis.

Suppose we have measured the response of a circuit to a sine wave so that we know

$$e^{i\omega t} \rightarrow a(\omega) e^{i\omega t}$$

where $a(\omega)$ is in general

complex. (Such a circuit is said to be linear). $|a|^2$ is the amplification factor. $\tan^{-1} \mathrm{Im}\, a / \mathrm{Re}\, a$ is the phase factor. The response of this circuit to the Fourier Transform of $f(x)$, $\phi(\omega)$, will be:

$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$\rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) a(\omega) e^{-i\omega x} dx = a(\omega)\phi(\omega)$$

We can construct the following table:

Input	Output
$e^{i\omega t} = e^{i\omega t}$	$a(\omega) e^{i\omega t} = a(\omega) e^{i\omega t}$
$1(t) e^{-\gamma t} = \frac{1}{2\pi}\int \frac{e^{i\omega t}}{\gamma + i\omega} d\omega$	$\frac{1}{2\pi}\int \frac{a(\omega) e^{i\omega t}}{\gamma + i\omega} d\omega =$ ...
$1(t) = \frac{1}{2\pi}\int \frac{e^{i\omega t}}{i\omega} d\omega$	$\frac{1}{2\pi}\int \frac{a(\omega) e^{i\omega t}}{i\omega} d\omega = \int_0^t R(t)\, dt$
$\delta(t) = \frac{1}{2\pi}\int e^{i\omega t} d\omega$	$\frac{1}{2\pi}\int a(\omega) e^{i\omega t} d\omega = R(t)$
$f(t) = \int \phi(\omega) e^{i\omega t} d\omega$	$\int \phi(\omega) a(\omega) e^{i\omega t} d\omega = \int_{-\infty}^{\infty} R(t-\tau) f(t)\, d\tau$

Math Methods

Lecture 39 Jan. 24, 1947

I Response of Linear Amplifiers

$$\delta(t) \rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} a(\omega) e^{i\omega t} d\omega \equiv R(t)$$

$$f(t) \rightarrow \int_{-\infty}^{\infty} a(\omega) \mathcal{F}(\omega) e^{i\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\omega) f(\tau) e^{i\omega(t-\tau)} d\omega\, d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) R(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} f(t-t') R(t') dt'$$

Response to δ function:

$$\delta(t) \rightarrow R(t)$$
$$\delta(t-a) \rightarrow R(t-a)$$
$$b\,\delta(t) \rightarrow b\,R(t)$$
$$\delta(t-a) + \delta(t-b) \rightarrow R(t-a) + R(t-b)$$

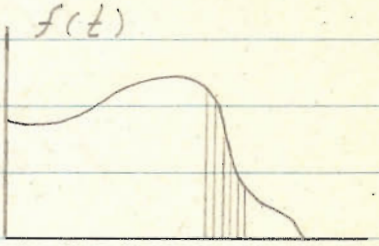
Suppose we represent a function as a series of δ function pulses with

appropriate coefficients. The response to one such pulse will be:

$$f(t')\,\delta(t-t') \rightarrow f(t')\,R(t-t')$$

$$f(t) = \int_{-\infty}^{\infty} f(t')\,\delta(t-t')\,dt'$$

$$= \int_{-\infty}^{\infty} f(t')\,R(t-t')\,dt'$$



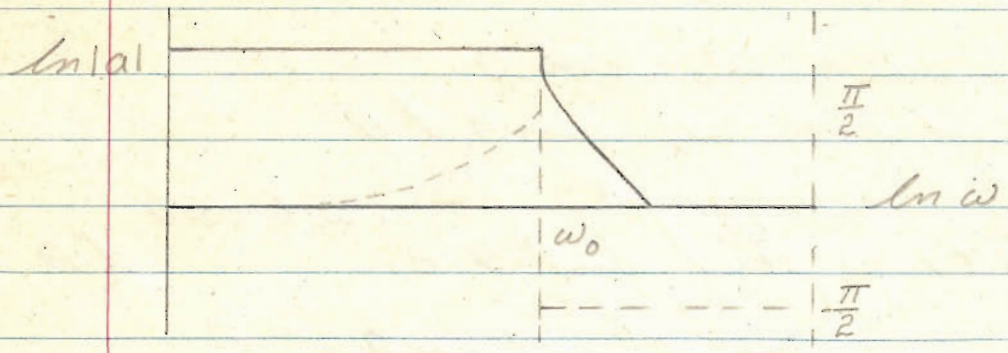
QED

Illustration Example:

Suppose for an amplifier:

$$a(\omega) = \left[\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2} + i\,\frac{\omega}{\omega_0}\right]^{-1}$$

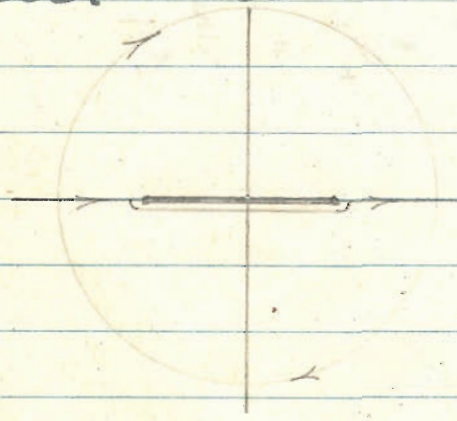
Amp $\lvert a(\omega)\rvert$	Phase
$\omega < \omega_0$ 1	$\tan^{-1} \dfrac{\omega/\omega_0}{\sqrt{1-\omega^2/\omega_0^2}}$
$\omega > \omega_0$ $\left[\sqrt{\frac{\omega^2}{\omega_0^2}-1} + \frac{\omega}{\omega_0}\right]^{-1}$ $\omega \gg \omega_0$ $\omega_0/2\omega$	$-\pi/2$



Let us find the response of this amplifier to a δ function

$$\delta(t) \rightarrow R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\left[\sqrt{1-\frac{\omega^2}{\omega_0^2}} - i\frac{\omega}{\omega_0}\right]d\omega$$

No poles but branch points at $\omega = \pm\omega_0$



For $t < 0$ the integral on the lower contour vanishes and

$$R(t) = 0 : t < 0$$

For $t > 0$ the integral on the upper contour vanishes and:

$$R(t) = \frac{2}{2\pi}\int_{-\omega}^{\omega}\sqrt{1-\frac{\omega^2}{\omega_0^2}}\, e^{i\omega t}\, d\omega$$

$$\omega = \omega_0 \cos\theta$$

$$d\omega = -\omega_0 \sin\theta \, d\theta$$

$$R(t) = \frac{\omega_0}{\pi} \int_0^{\pi} \sin^2\theta \, e^{i\omega_0 t \cos\theta} \, d\theta$$

$$= \frac{\omega_0}{2\pi} \int_0^{\pi} e^{i\omega_0 t \cos\theta} (1 - \cos 2\theta) \, d\theta$$

$$= \frac{\omega_0}{2\pi} \frac{1}{2} \left(2\pi J_0(\omega_0 t) - 2\pi i^2 J_2(\omega_0 t) \right)$$

$$= \frac{\omega_0}{\omega_0 t} J_1(\omega_0 t)$$

$$= \frac{1}{t} J_1(\omega_0 t)$$

$$J_0 + J_2 = \frac{2}{V} J_1$$

Math Methods

Lecture 10 Jan 27, 1947

I Application of Fourier Trans. to solution of the heat equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

$$T(xt) = \int_{-\infty}^{\infty} \psi(\omega t) e^{i\omega x} d\omega$$

$$-\omega^2 \psi = \frac{\partial \psi}{\partial t}$$

$$\psi = \phi(\omega) e^{-\omega^2 t}$$

$$T(x0) = \int_{-\infty}^{\infty} \phi(\omega) e^{i\omega x} d\omega$$

$$\phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(x0) e^{-i\omega x} dx$$

$$T(xt) = \frac{1}{2\pi} \int_{-\omega}^{\infty} \int_{-\infty}^{\infty} T(x'0) e^{i\omega(x-x')-\omega^2 t} dx' d\omega$$

$$= \int_{-\infty}^{\infty} \frac{T(x'0)}{\sqrt{4\pi t}} e^{-\frac{(x-x')^2}{4t}} dx'$$

II Double Fourier Transforms

$$Q(xt) = \int e^{i\omega t} g'(\omega x)\, d\omega$$

$$g'(\omega x) = \int e^{ikx} g(\omega k)\, dk$$

$$Q(xt) = \iint g(\omega k) e^{i\omega t} e^{ikx}\, d\omega\, dk$$

$$g(\omega k) = \frac{1}{4\pi^2} \iint Q(xt) e^{-i\omega t - ikx}\, dx\, dt$$

Solution of heat equation

$$\frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial t} = -Q(xt)$$

Transformed equation

$$(-k^2 - i\omega) T = -g$$

$$T(xt) = \iint \frac{g(\omega k)}{k^2 + i\omega} e^{i\omega t} e^{ikx}\, d\omega\, dk$$

$$T(xt) = \frac{1}{4\pi^2} \iiint\!\!\int \frac{Q(x't')}{k^2 + i\omega} e^{i\omega(t-t')} e^{ik(x-x')}\, d\omega\, dk\, dx\, dt$$

Integral on ω

$$\int_{-\infty}^{\infty} \frac{e^{i\omega z}}{k^2 + i\omega}\, d\omega = e^{-k^2 z}\, 1(z)\, 2\pi$$

$$T(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{t} Q(x',t') e^{-k^2(t-t')} e^{ik(x-x')} \, dk \, dt' dx'$$

Integral on k

$$\int_{-\infty}^{\infty} e^{-k^2 z} e^{ikz'} \, dk = \sqrt{\frac{\pi}{z}} e^{-\frac{z'^2}{4z}}$$

$$T(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{t} \frac{Q(x',t')}{\sqrt{4\pi(t-t')}} e^{-\frac{(x-x')^2}{4(t-t')}} \, dx' dt'$$

This result is the same as that obtained by Green's method.

Math Methods

Lecture 41 Jan 29, 1947

I Laplace Transforms

Ref. Bell Tel. Tech. Journal
#7 p 639 1928

Laplace Transforms are almost exactly like Fourier Transforms. They are used mostly in circuit analysis because of their convenience They assume that $f(t) = 0$ for $t < 0$. The transformation from Fourier Transforms is effected by $i\omega \to p$
$\phi(\omega) \to \Phi(p)/2\pi$

$$\Phi(p) = \int_0^\infty f(t) e^{-pt} dt$$

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi(p) e^{pt} dp$$

All the singularities of $\Phi(p)$ lie to the left of the imaginary axis. c is a real number greater than zero

II Convolution:

$$g(t) = \int_{-\infty}^{\infty} \mathcal{G}(\omega) e^{i\omega t} d\omega$$

$$f(t) = \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{i\omega t} d\omega$$

$$\int_{-\infty}^{\infty} \mathcal{G}(\omega) \mathcal{F}(\omega) e^{i\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t') g(t - t') dt'$$