Math. 7a.

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NATIONAL

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THIS BOOK CONTAINS EYE-EASE PAPER

"Easy on the Eyes"

Math Methods

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Lecture I 2/17/47

I Differentiation of a Product

$$\frac{d}{dx}\ln(u^a v^b \dots) = \frac{d}{dx}(u^a v^b \dots)/(u^a v^b \dots)$$

$$= \frac{a}{u}\frac{du}{dx} + \frac{b}{v}\frac{dv}{dx} + \dots$$

$$\frac{d}{dx}(u^a v^b \dots) = (u^a v^b \dots)\left(\frac{a}{u}\frac{du}{dx} + \frac{b}{v}\frac{dv}{dx} + \dots\right)$$

II Partial Differential

Dealing with derivatives is mostly a matter of messing around to get desired results. A few formulas that it is useful to remember are the following:

$u(x,y)$ $\quad du = \left.\frac{\partial u}{\partial x}\right|_y dx + \left.\frac{\partial u}{\partial y}\right|_x dy$

$$\left.\frac{\partial u}{\partial x}\right|_y = 1 \Big/ \left.\frac{\partial x}{\partial u}\right|_y$$

Suppose also $x(\xi\eta)$ & $y(\xi\eta)$

$$\left.\frac{\partial u}{\partial \xi}\right. = \left.\frac{\partial u}{\partial x}\right|_y \left.\frac{\partial x}{\partial \xi}\right|_\eta + \left.\frac{\partial u}{\partial y}\right|_x \left.\frac{\partial y}{\partial \xi}\right|_\eta$$

$$1 = \left.\frac{\partial \xi}{\partial x}\right|_y \left.\frac{\partial x}{\partial \xi}\right|_\eta + \left.\frac{\partial \xi}{\partial y}\right|_x \left.\frac{\partial y}{\partial \xi}\right|_\eta$$

$$0 = \left.\frac{\partial \eta}{\partial x}\right|_y \left.\frac{\partial x}{\partial \xi}\right|_\eta + \left.\frac{\partial \eta}{\partial y}\right|_x \left.\frac{\partial y}{\partial \xi}\right|_\eta$$

Problem: Find in spherical coordinates.

$$\nabla u \; ; \; \nabla^2 u \; ; \; y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y}$$

III Transformation of volume elements may be accomplished by means of the Jacobian determinant

$$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} d\eta\, d\xi$$

IV Differential Equations in order of difficulty

1. Linear with constant coefficients - Solved!
2. Linear - the lower the order the better.
3. Non-linear - the lower the order the better.

Linear equations are characterized by the fact that if u and v are two solutions of the equation, then $u + v$ is also a solution.

Methods of Solution

1) Suggestions that might work e.g. substitution of a new variable

2) Definite Methods applicable to certain types of equations

31 Approximate methods
a. Series b. Arithmetic

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I Methods to reduce the order of a differential equation. A differential equation of the n^{th} order has a solution containing n arbitrary constants. If the position of one of these arbitrary constants can be determined, it is possible to reduce the equation to one of the next lower order. All elementary methods of reducing the order of differential equations depend on this fact.

Below are methods for solving a few special types of

differential equations

a. Equations in which y does not appear explicitely. Only $y' y'' \ldots + x$. In this case if $y = f(x)$ a solution, then $y^\circ = f(x) + c$ also a solution. The equation can be reduced by the substitution:

$$z = y' \; ; \; x = x$$

b. Equations in which x does not appear explicitely. Only $y\ y'\ y'' \ldots$. In this case if $y = f(x)$ a solution, then $y^\circ = f(x+c)$ also a solution. The equation can be reduced by the substitution

$$p = y' \qquad y = y$$

Equations of this type arise in mechanics where the potential

does not depend on time and hence t does not appear explicitly in the equations

$$\ddot{y} = \frac{dp}{dt} = \frac{dp}{dy}\frac{dy}{dt} = p\frac{dp}{dy} = \frac{d}{dy}\left(\frac{1}{2}p^2\right)$$

c. Homogenous Equations. If increasing x by a factor c and y by a factor c^n leaves the equation unchanged, then if $y = f(x)$ a solution, $c^n y = f(cx)$ also a solution. The equation may be reduced by the substitution:

$$z = y/x^n \qquad u = x\frac{d}{dx}\left(\frac{y}{x^n}\right)$$

This is best accomplished by

$y = zx^n$ followed by

$v = \ln x$ which reduces the equation to case b.

Example: $\frac{d^2y}{dx^2} = \frac{x}{\sqrt{y}}$ to determine $x \to cx'$

$y \to \beta y'$

$$\frac{\beta}{c^2}\frac{d^2y'}{dx'^2} = \frac{cx'}{\sqrt{\beta}\sqrt{y'}}$$

The equation will be the same as the original equation if

$$\frac{\beta^{3/2}}{c^3} = 1$$

$$\beta = c^2$$

For this case the substitution

$$y = zx^2 \quad \text{is appropriate}$$

$$y'' = z''x^2 + 2z'2x + 2z$$

$$x'/\sqrt{y} = x/\sqrt{zx^2} = 1/\sqrt{z}$$

Now

$$x = e^{v}$$

$$z' = e^{-v}\frac{dz}{dv}$$

$$z'' = e^{-2v}\frac{d^2z}{dv^2} - e^{-2v}\frac{dz}{dv}$$

Hence $$\frac{d^2z}{dv^2} + 3\frac{dz}{dv} + 2z = 1/\sqrt{z}$$

This is case b. $$\frac{dz}{dv} = p$$

$$\frac{d^2z}{dv^2} = p\frac{dp}{dz}$$

$$2z + 3p + p\frac{dp}{dz} = 1/\sqrt{z}$$

d. Equations in which an increase in y by a factor c leaves the equation unchanged. In this case if $y = f(x)$, then $cy = f(x)$ also a solution. The equation can be reduced by the substitution

$$z = \frac{1}{y}\frac{dy}{dx} \; ; \quad x = x$$

but accomplished by $v = \ln y$ which reduces the equation to case a.

II First Order Diff. Equation

$$\frac{dy}{dx} = f(xy)$$

This equation is not soluble in the general case.
Soluble cases:

a. Linear in y.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} + P(x)y = y^n Q(x)$$

$$\frac{1}{y^n} y' + P(x) y^{1-n} = Q(x)$$

$$\frac{1}{(1-n)} \frac{dy^{1-n}}{dx} + P(x) y^{1-n} = Q(x)$$

b. Homogeneous.

$$f(\alpha x, \beta y) = \alpha^r \beta^s f(x,y)$$

$$\frac{dy}{dx} = f(x,y)$$

$$\frac{\beta}{\alpha} \frac{dy'}{dx'} = \alpha^r \beta^s f(x',y')$$

The equation is unchanged if

$$\alpha^r \beta^s \frac{\alpha}{\beta} = 1 \; : \; \beta = \alpha^{\frac{r+1}{1-s}}$$

This may be reduced by I.-c

c. Separable.

d. $M(xy)\,dx + N(xy)\,dy = 0$

Soluble if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = 0$$

$$\phi = \int M\,dx + F_1(y) = \text{const.}$$
$$\phi = F_2(x) + \int N\,dy = \text{const.}$$

e. Forms involving $x\,dy - y\,dx$ have integrating factors

$1/(x^2+y^2)$, x/y, y/x, $1/x^2$, $1/y^2$

f. Variable Transformations

g. Put equation in form

$$F(x\,y\,p) = 0$$

$$p = \frac{dy}{dx}$$

Problems: $x, y, p = \frac{dy}{dx}$

1. $(1+x)y^2dx - x^3dy = 0$ sep

2. $p - \frac{y}{x+1} = x$ lin

3. $p = \frac{1}{(y-x)^2}$ $v = y - x$

4. $xp + y + x^4y^4e^x = 0$ lin

5. $xdy - ydx = \sqrt{x^2+y^2}dx$ C.

6. $y^2(1+p^2) = a^2$ sep

7. $(px+y)(py+x) = a^2p$ $x+y$ $x-y$

8. $(y - xp)/\sqrt{1+p^2} = x^2+y^2$ polar

9. $2px - y + \ln p = 0$ diff. - C.

10. $p^2 + 2xp - y = 0$ C.

Lecture 3 2/21/47

I Approximate Methods of solving differential Equations

a. Integration of Series
b. Asymptotic Estimates
c. Iteration
d. Arithmetic

To illustrate these methods we shall use the synchrotron equation:

$$y'' = x - y^2$$

a. Series

Assume the initial conditions $y = 1$, $y' = 0$ for $x = 0$. We want a solution for $x > 0$

$$y = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \dots$$
$$y' = 0 + b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 + \dots$$
$$y'' = 0 + 0 + 2c + 6dx + 12ex^2 + 20fx^3$$

We can simplify immediately by inverting our initial conditions:

$$y = 1 + cx^2 + dx^3 + ex^4 + fx^5 + \dots$$
$$y^2 = 1 + x^2(2c) + x^3(2d) + x^4(2e + c^2) + x^5(2f + 2cd) + x^6(2g + 2ce + d^2)$$

Hence

$$2c + 6dx + 12ex^2 + 20fx^3 + 30gx^4 + \dots$$

$$= x - 1 - 2cx^2 - 2dx^3 - (2e + c^2)x^4 + \dots$$

Comparing coefficients

$$c = -1/2$$
$$d = 1/6$$
$$e = 1/12$$
$$f = -1/60$$
$$g = -1/72 \quad \text{etc.}$$

$$y = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{12}x^4 - \tfrac{1}{60}x^5 - \tfrac{1}{72}x^6$$

This converges for x not too much greater than 1. For $x > 1$, one way of getting a solution is to use the series found to compute initial conditions at say .9 and then expand in a new series about .9. In general one expands about the point for which the initial conditions are known. If the initial conditions are given in the form $y = 1$ for $x = 0$ and x_1, the only way to solve the problem by this method is to try different values of y' at $x = 0$ until the curve goes thru the point $(x_1, 1)$

Problem: Using series solve $y'' = x - y^2$ for $y = \frac{1}{2}$ $y' = \frac{1}{2}$ at $x = 0$

Problem: Using series solve

$$y' = e^{-2xy} \qquad y = 0 \text{ at } x = 0$$

6 Asymtotic Estimates

$$y'' = x - y^2$$

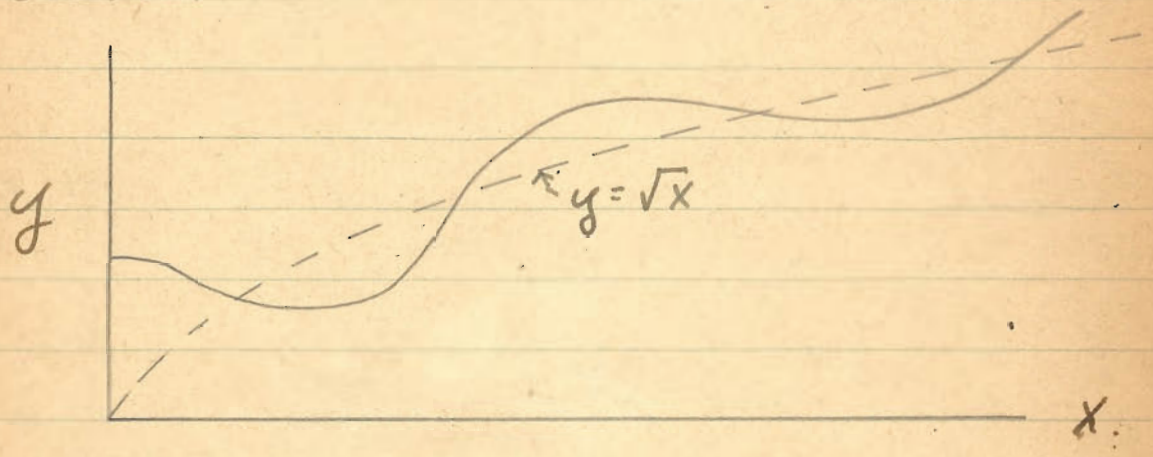
Suppose for large x $y^2 << x$

$$y'' = x$$
$$y = \frac{1}{6} x^3$$

This is inconsistant with our initial assumption. ∴ n.g.
We could try $y^2 >> x$ for large x ~~but this leads to an elliptic integral.~~

By examining the differential equation, we find that if $y^2 > x$, the solution, y, must be concave toward the x axis and if $y^2 < x$, y

convex toward the x axis. Hence we might look for a solution of form illustrated in the graph below:



To a first approximation, we may write $y \sim \sqrt{x}$. As a next approximation, let us try:

$$y = \sqrt{x} + \eta$$

$$-\frac{1}{4x^{\frac{3}{2}}} + \frac{d^2\eta}{dx^2} = x - (\sqrt{x} + \eta)^2$$

$$= -2\sqrt{x}\,\eta - \eta^2$$

For large x, assume η^2 and

$-\frac{1}{4x^{\frac{3}{2}}}$ small. Therefore:

$$\frac{d^2\eta}{dx^2} \simeq -2\sqrt{x}\,\eta$$

In connection with this equation let us consider the general equation:

$$\frac{d^2\eta}{dx^2} = -f(x)\eta \qquad f(x) > 0$$

We can obtain an approximate solution of this equation - under certain conditions - by use of the WKB method. Try

$$\eta = e^{iA(x)}$$

$A(x)$ a function to be determined.

$$\dot{\eta} = i\dot{A}e^{iA}$$
$$\ddot{\eta} = -\dot{A}^2 e^{iA} + i\ddot{A}e^{iA}$$

Assume $|\ddot{A}| << |\dot{A}^2|$

$$-\dot{A}^2 e^{iA} = -f(x)e^{iA}$$

$$\dot{A}(x) = \sqrt{f(x)}$$

$$A_0 = \int^x \sqrt{f(y)}\, dy$$

Hence in first approx

$$\eta = e^{i \int^x \sqrt{f(y)}\, dy}$$

Let us check to see under what conditions our assumption, $|\ddot{A}| << |\dot{A}^2|$, is valid.

$$|\ddot{A}| = \frac{f'}{2\sqrt{f}} << |\dot{A}^2| = f \quad ; \quad f > 0$$

$$\frac{1}{2} f'/f/\sqrt{f} << 1$$

To a next approximation let us try the solution

$$\eta = e^{i(A_0 + A_1)}$$

$$\dot{\eta} = i(\dot{A}_0 + \dot{A}_1)\, e^{i(A_0 + A_1)}$$

$$\ddot{\eta} = (i\ddot{A}_0 + i\ddot{A}_1 - \dot{A}_0^2 - 2\dot{A}_0\dot{A}_1 - \dot{A}_1^2)\eta = -f(x)\eta$$

Assume $\dot{A}_1^2$ and $\ddot{A}_1$ small. By definition $\dot{A}_0^2 = f(x)$

$$2\dot{A}_1\dot{A}_0 = i\ddot{A}_0$$

$$\dot{A}_1 = \frac{i}{2}\ln \dot{A}_0 + c$$

$$\eta = Be^{iA_0} e^{-\frac{1}{2}\ln \dot{A}_0}$$

$$* \quad \underline{\eta = \frac{B}{\sqrt[4]{f}} e^{i\int_c^x \sqrt{f(y)}\,dy}}$$

Problem: $m\ddot{x} = -kx \qquad k(t)$

Problem: Describe for large $x+y$.

$$\frac{d}{dx}\left(x\frac{d\psi}{dx}\right) = -\psi$$

* True if: $|1 - \dot{A}_1^2 + i\ddot{A}_1| < |i\ddot{A}_0|$

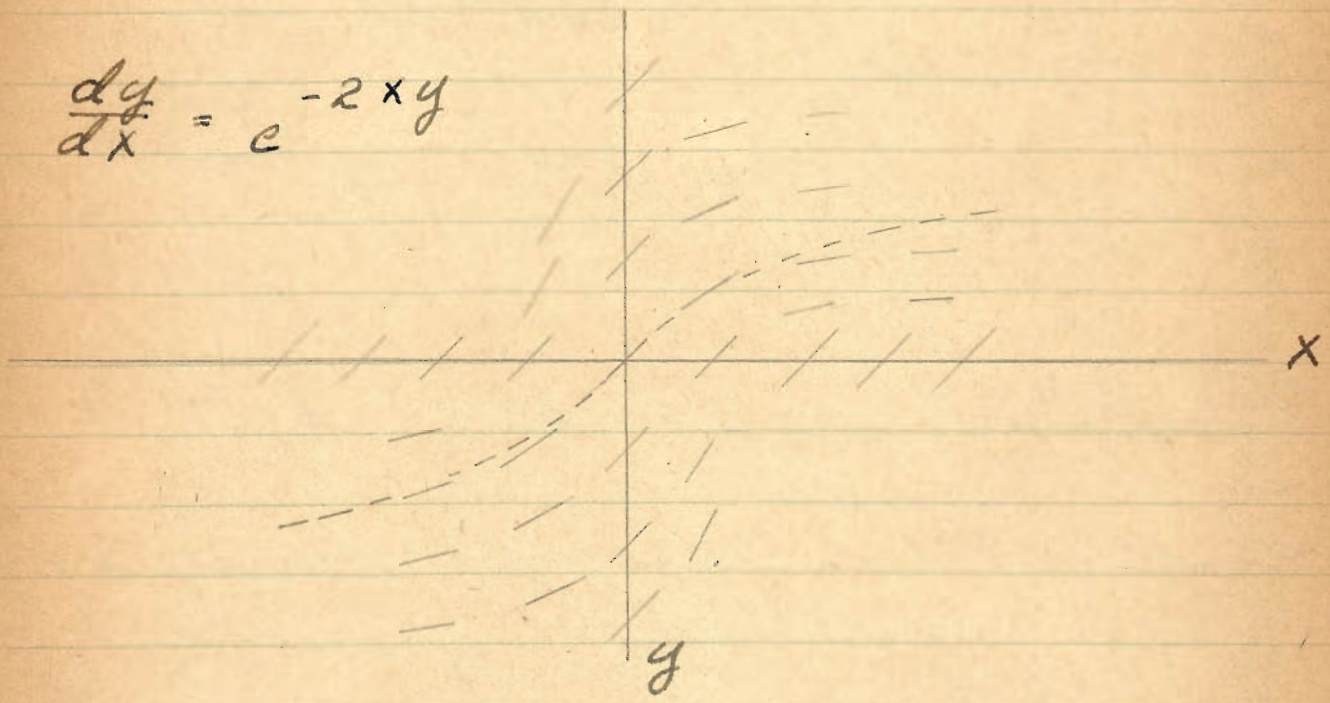
$$\frac{5}{16}\left(\frac{f'}{f}\right)^2 - \frac{1}{4}\frac{f''}{f} << \frac{f'}{2\sqrt{f}} << f$$

$$\frac{5}{16}\left(\frac{f'}{f}\right)^2\frac{1}{f} - \frac{1}{4}\frac{f''}{f^2} << 1$$

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We can get an approximate idea of the behavior of the solution of a differential equation by making a

$$\frac{dy}{dx} = e^{-2xy}$$



Iteration:

$$\frac{dy}{dx} = e^{-2xy}$$

From the above diagram we see that for large x

y is approaching a constant.
As a first approximation
try $y_0 = a$ (for large x)

2nd $\quad \frac{dy_1}{dx} = e^{-2xa}$

$$y_1 = a - \frac{1}{2a} e^{-2xa}$$

3rd $\quad \frac{dy_2}{dx} = e^{-2x\left(a - \frac{1}{2a} e^{-2xa}\right)}$

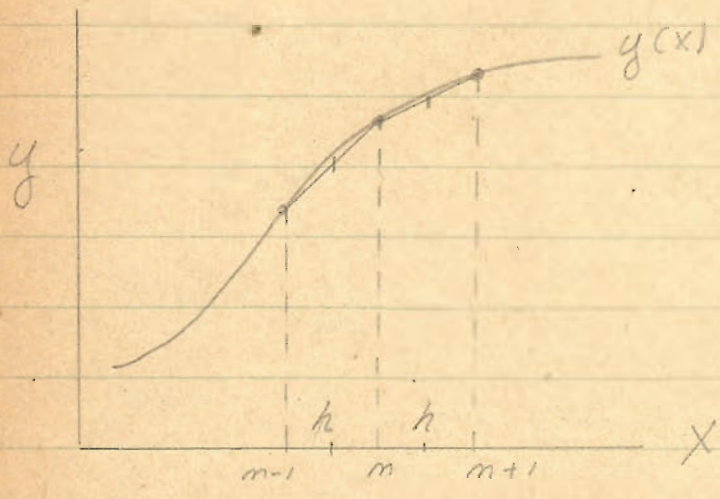
$$= e^{-2xa}\left(1 + \frac{x}{a} e^{-2xa} + \dots\right)$$

$$y_2 = a - \frac{1}{2a} e^{-2xa} - \frac{1}{16a^3}(1+4xa) e^{-4xa} \dots$$

This solution is good for large x. As a problem we found a series solution of this equation for small x. By fitting the two solutions at intermediate x, we obtain a solution for all x.

Problem: Find an approximate solution of the equation $y' = e^{-2xy}$ for all x. $\quad y = 0, y' = 1$ for $x = 0$

Numerical Solution of differential equations.



It is clear from the above diagram that

$$y'_{n-\frac{1}{2}} \sim (y_n - y_{n-1})/h$$
$$y'_{n+\frac{1}{2}} \sim (y_{n+1} - y_n)/h$$

and

$$y''_n \sim (y'_{n+\frac{1}{2}} - y'_{n-\frac{1}{2}})/h$$

These relations are most conviently used in the form:

$$h\,y'_{n+\frac{1}{2}} \sim y_{n+1} - y_n$$
$$h^2 y''_n \sim (y'_{n+\frac{1}{2}} - y'_{n-\frac{1}{2}})h$$

Consider the equation

$$y'' = x - y^2$$

To get started, we shall need two values of y. These may be obtained from the series solution of the problem for small x. It is better to use values of y_0 and y_1 to get started than values of y_0 and $y'_{\frac{1}{2}}$. Errors in the method tend to cancel better in the first case. We have $y = 1, y' = 0$ at $x = 0$. Our series was $y = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$

Let $h = .2$ $\quad y(.2) = .9814$

x	y	$.2\,y'$	$.04\,y''$	
0	1.0000			
		.0186		
.2	.9814	→	.0305	$= .04(x - y^2)$
	← .0491		←	
.4	.9323	→	.0188	
	← .0679		←	
.6	.8644	→	...	
.8	...	...		
1.0				
1.2				

Check smoothness by taking differences last column ($.04y''$)
Check against series or asymtotic estimate where possible.

Problem: Solve numerically, for x 0 to 2, $y'' = x - y^2$, for $y = \frac{1}{2}$, $y' = \frac{1}{2}$ at $x = 0$.

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I Errors in Numerical Methods

a) Systematic Errors

In using the difference formula to compute the second derivative we make an error:

$$h^2 y_0'' - (y_{+1} - 2y_0 + y_{-1}) = \text{error in last column.}$$

To find y_{+1} & y_{-1} expand y in a Taylor series about y_0

$$y_{+1} = y(x+h) = y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \frac{h^4}{24}y^{IV}$$
$$y_{-1} = y(x-h) = y - hy' + \frac{h^2}{2}y'' - \frac{h^3}{6}y''' + \frac{h^4}{24}y^{IV}$$

$$y_{+1} + y_{-1} - 2y = h^2 y'' + \frac{h^4}{12}y^{IV}$$

The error in h^2y'' is $\frac{h^4}{12}y^{IV}$. which is approximately 1/12 the second difference in the tabulated values of h^2y''. The systematic error ought to appear in the last figure of the values of h^2y''. The answer is obtained by dropping the last two figures (depends on the number of intervals but for < 30 the above rule is about right.) Carrying two few figures or using too small an interval may lead to errors if y'' turns out zero. Enough figures should be carried to give a value of y''. This is what causes the function to

change.

b) Accidental Error.

Consider the equation $h^2y'' = h^2(x^2 - y^2)$. This is the formula used to compute the values of h^2y''. Suppose there is an error of ϵ in y, then the error in h^2y'' will be

$$h^2y'' = h^2(x - (y+\epsilon)^2)$$
$$= h^2(x - y^2) - h^2 2y\epsilon$$

Error in $h^2y'' = -\underline{2yh^2\epsilon}$

If $|2yh^2|$ is large the error in the answer may accumulate very rapidly. The method should only be applied where $|2yh^2| < 1$. In our solution we used $h = .2$, hence $.08y^2 < 1$ and $y < 12.5$

The method of numerical

integration outlined above is known as Hartrees methods. It is applicable to equations of the following types

$$\frac{d^2y}{dx^2} = f(xy)$$

$$\frac{d}{dx}\left[F(x)\frac{dy}{dx}\right] = g(xy)$$

Let $dx/dz = F(x)$ reduces equation to:

$$\frac{d^2y}{dz^2} = g'(zy)F'(z)$$

However, in general, it is better to solve this equation in the original form.

x	y	$h y'$	$h F(x) y'$	$h^2 \frac{d}{dx} F(x) y'$	$= h^2 g(xy)$
A	B	$B' - B$	$F(x)[B'-B]$		
A'	B'	$" + \frac{c}{F}$	$" + c$	c	
A''					

The method does not work on

equations involving $\frac{dy}{dx}$ since the value of this quantity is not known at the same point as the values of y, x, and y''.

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Numerical Integration of First order differential equations.
The two most obvious methods of solving the equation $y' = f(x,y)$ numerically would be:

a. $y_{n+1} = y_n + h y'_n$

b. $y_{n+2} = y_n + 2h y'_{n+1}$

The first of these methods is too inaccurate, and the second is subject to a large accidental error.

The best way of solving a first order differential

equation numerically is to use the two relations

a. $y_{n+1} = y_n + \frac{h}{2}(y'_n + y'_{n+1})$

b. $y'_{n+1} = f(x_{n+1}, y_{n+1})$

which may be combined to give:

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

If x_n, x_{n+1}, and y_n are known, we can solve for y_{n+1} by trial and error.

As an illustration consider the equation $y' = x + y$, $y(0) = 1$

x	y	$.2(x+y)$
.0	1.000	.200
.2	$(1.200)_1$	.280
	$(1.240)_2$	.288
	$(1.244)_3$	.289
	1.244	.289
.4	...	...

Better accuracy is obtainable with about the same amount of work if we use Simpsons rule. The formula for computing y_{n+1} is:

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

To get started we need two values of y. These are best obtained by series. The interval h may be chosen larger than if the trapazoidal rule is used. For ease of calculation h should be divisible by 3.

To aid in making our guesses we may use Milnes rule for integration:

$$(y_{n+1})_{guess} = y_{n-3} + \frac{4h}{3}(2y'_n - y'_{n-1} + 2y_{n-2})$$

In using Milnes rule it may

not be necessary to check every guess, but guesses should be checked every third or fourth interval. The systematic error in this method is

$$\frac{1}{29}\left(y_{\text{guess}}^{\text{Milne}} - y_{\text{true}}^{\text{Simpson}}\right)$$

Equations of the form:

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + \tan x\, y$$

can be solved by combining the above methods.

Problem: Solve: $y' = e^{-2xy}$
$y(0) = 0$ for $x = 0$ to 1.8 using $h = .3$

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I Solution of Partial Differential Equations

a. Separation Variables

Heat Flow:

$$\frac{\partial T}{\partial t} = \nabla^2 T$$

Let $T = f(t)\,\phi(xyz)$

$$\phi f' = f \nabla^2 \phi$$

$$\frac{f'}{f} = \frac{\nabla^2 \phi}{\phi} = K$$

$$f = e^{Kt}$$

$$\nabla^2 \phi = K \phi$$

Wave Equation:

$$-\nabla^2 \psi + (E - V)\psi = 0$$

Separate:

$$\psi = F(x)\,G(yz)$$

$$\psi = R(r)\,Y(\theta\phi)$$

Illustrative Equation:

$$\frac{\partial \phi^2}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

Let: $\phi = F(t)\,G(x)$

$$G^2 2FF' = FG''$$

$$2F' = G''/G^2 = 2K$$

$$\rightarrow \quad F = a + Kt$$

$$G'' = 2KG^2$$

$$G' = \left(b + \frac{4K}{3}G^3\right)^{\frac{1}{2}}$$

$$\rightarrow \quad x = \int dG \Big/ \sqrt{b + \frac{4K}{3}G^3}$$

Problem: In a star the temperature satisfies the equation:

$$\frac{\partial}{\partial x}\left(T^k \frac{\partial T}{\partial x}\right) = \frac{\partial T^4}{\partial t} \qquad k = 7$$

1) Indicate how to solve in the form $T = f(t)\,g(x)$

2) Given a rod in which $T_{x=0}$

varies as the m^{th} power of the time, Find the temperature as a function of x & t.

b. Method of Similarity

Consider the equation:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

We note that if we change x by a factor α, and t by a factor α^2 this leaves the equation unchanged. It is possible therefore that there may be a solution of the equation which also possesses this property:

$$F(y) = F\left(\frac{x}{\sqrt{t}}\right) = T.$$

$$-\frac{x}{2t^{\frac{3}{2}}} F' = \frac{1}{t} F''$$

$$-\frac{1}{2} y F' = F''$$

$$F' = e^{-\frac{x^2}{4t}} = T'$$

$$F = \int e^{-\frac{y^2}{4}} dy = T.$$

We note that at $x = 0$, $t \neq 0$ $T(0)$ = a constant, C_1. For $x = 0$, $t = 0$, $T(\frac{0}{0})$ may $= C_2$. This form of solution is good for solving heat flow problems in which a temperature step is put in at time 0.

In general, we must find a solution of the differential equation which will satisfy the boundary conditions.

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I Solution of Partial Differential Equations by Similarity (cont.)

Consider the equation:

$$\frac{\partial \phi^2}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

Transform

$$\phi = \beta \phi_1$$
$$x = \alpha x_1$$
$$t = \gamma t_1$$

The equation becomes:

$$\frac{\beta^2}{\gamma} \frac{\partial \phi_1^2}{\partial t_1} = \frac{\beta}{\alpha^2} \frac{\partial^2 \phi}{\partial x^2}$$

Therefore the equation invariant if: $\beta = \gamma / \alpha^2$. Consider a new variable $y = x t^k$ and fix $\alpha \gamma^{-k} = 1$, therefore $\beta = \gamma^{2k+1}$ & ϕ of the form:

$$\phi = t^{2k+1} F(x t^k)$$

has the desired properties for invariance. Substituting in the differential equation, we obtain:

$$(4k+2) t^{4k+1} F^2 + 2 t^{4k+1} F F' x k t^{k-1}$$

$$= t^{2k+1} t^{2k} F''$$

$$(4k+2)F^2 + 2kyFF' - F'' = 0$$

We have reduced the problem to the solution of a total differential equation.

In general, to solve problems by this method substitute:

$$\phi = t^m F(xt^n)$$

If values of m & n can be found such that the differential equations contains only $F(xt^n)$ its derivatives and xt^n in any combination, the method of solution is applicable.

Special cases of the above substitution are:

1:

$$\phi = e^{2ct} F(xe^{ct})$$

2:

$$\phi = t\,G(te^{dx})$$

This method of solution solves

problems in which the boundary conditions at $x = 0$ are:

a: $\phi \sim t^n$

b: $\phi \sim e^{2ct}$

c: $\phi \sim t\,G(t)$

II Linear Equations:

Let L represent a differential operator. L is linear if $L(u+v) = Lu + Lv$ and $L(au) = aL(u)$. Linear differential equations can be represented in the form:

$$Lu = 0$$

$$Lu = X(x)$$

where L is a linear operator.

A. First order equations

$$u' + P(x)u = Q(x)$$

Solved!

B. Second order equations

$$u'' + P(x)u' + Q(x)u = X(x)$$

We have already solved the following 2nd order linear equations:

1: $u'' + k^2 u = 0 \qquad u = Ae^{\pm kx}$

2: $u'' + \frac{1}{x}u' + (1 - \frac{n^2}{x^2})u = 0 \qquad u = J_n, N_n$

3: $u'' - 2xu' + 2mu = 0 \qquad u = H_m$

We can transform a second order linear equation into another second order equation as follows. Consider

$$y'' + py' + qy = X(x)$$

Let $y = f(x)V$ The equation becomes:

$$v'' + P_1 v' + Q_1 v = X_1(x)$$

where:
$$P_1 = \frac{2f'}{f} + P$$
$$Q_1 = f''/f + f'/f\,P + Q$$
$$X_1 = X/f$$

The above transformation has a special use. Suppose one solution of the homogeneous equation $Lg = 0$ is known: $g(x)$. Choose $f = g(x)$. Q_1 becomes

$$Q_1 = (g'' + Pg' + Qg)/g = 0$$

$\therefore$

$$v'' + P_1(x) v' = X_1$$

Let $p = v'$

$$p' + P_1(x) p = X_1 \quad \text{Done!}$$

Problem: Solve $L(y) = X(x)$, given that $Lg = 0$.

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I Transformation of Linear Second order differential equations continued.

$$y'' + Py' + Qy = X$$

To eliminate the first order term choose $y = V e^{-\frac{1}{2}\int^x P dy}$. The equation becomes:

$$V'' + (Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2)V = X e^{\frac{1}{2}\int^x P dy}$$

This equation has been solved for the following coefficients of V

a.	X^n	Bessel
b.	$a - bx^2$	Hermite
c.	$c - dX$	$J_{1/3}$
d.	$a - b\cos X$	Mathieu
e.	$a - b/X$	
f.	$a - b/X - c/X^2$	

Second method of transforming equations.

$$y'' + P(x)y' + Q(x)y = X(x)$$

Substitute $z = g(x)$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

$$\frac{d^2x}{dx^2} = \frac{dy}{dz}\frac{d^2z}{dx^2} + \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2$$

The equation becomes

$$\frac{d^2y}{dz^2} + \left(\frac{g'' + g'P}{g'^2}\right)\frac{dy}{dz} + \left(\frac{Q}{g'^2}\right)y = \frac{X(x)}{(g')^2}$$

Try $g' = \sqrt{Q}$. It may make the coefficient of $\frac{dy}{dz}$ a constant in which case the equation can be solved.

To make the coefficient of $\frac{dy}{dz}$ zero

$$g'' = -g'P, \quad g' = e^{-\int^x P(x')dx'}$$

The equation becomes:

$$\frac{d^2y}{dz^2} + Qe^{2\int^x P(x')dx'} y = Xe^{2\int^x P(x')dx'}$$

Problem: Eliminate the middle term in the equation $u'' - 2Xu' + 2mu = 0$ by two methods.

Consider the equation $y'' + Qy = 0$
This equation may be transformed by substituting

$$y = f(x)v$$
$$z = \int^x dx' / f(x')^2$$

The transformed equation is

$$v'' + f^3(f'' + Qf)v = 0$$

II Properties of solutions of differential equations.

A. Eigenvalue problem.

The eigenvalue problem consists of determining solutions of the

equation $Lu = \lambda u$ where u is subject to certain boundary conditions. It will be found that there are solutions for only certain values of the constant λ.

As an illustration consider the equation $u'' = -\omega^2 u$ where $u = 0$ for $x = 0$ and l. The general solution is $u = A \sin \omega x + B \cos \omega x$. Applying our boundary conditions we find solutions for $B = 0$ and $\omega^2 = n^2 \pi^2 / l^2$. The values of ω^2 for which there are solutions are called eigenvalues.

Other examples of eiginvalue problems: $\frac{1}{\sin\theta} \frac{d}{d\theta}(\sin\theta \frac{du}{d\theta}) = \lambda u$ subject to the condition that u be finite everywhere has solutions for $\lambda = n(n+1)$ which are $P_n(\cos\theta)$.

$u'' + x^2 u = \lambda u$ subject to the condition that $u = 0$ at $\pm \infty$

has solutions for $\lambda = 2(n+\frac{1}{2})$ which are $e^{-\frac{x^2}{2}} H_n(x)$

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I Eigenvalue Problem. We are looking for solutions of the equation $Lu = \lambda u$ which satisfy certain boundary conditions. For certain values of the paramter λ solutions will exist. The function u_i corresponding to a given value of the paramter, λ_i is called the eigenfunction belonging to the eigenvalue λ_i.

Consider the two equations

u_j // $\quad u_i'' + Q u_i = \lambda_i u_i$

u_i // $\quad u_j'' + Q u_j = \lambda_j u_j$

Multiplying the first eq. by u_j and the second by u_i and

subtracting to eliminate Q, we obtain:

$$u_j u_i'' - u_i u_j'' = (\lambda_i - \lambda_j) u_i u_j$$

$$\frac{d}{dx}(u_j u_i' - u_i u_j') = (\lambda_i - \lambda_j) u_i u_j$$

Integrating

$$(u_j u_i' - u_i u_j')\Big|_a^b = (\lambda_i - \lambda_j) \int_a^b u_i u_j$$

Suppose we have the boundary condition that at a & b u or u' must be zero. We obtain

$$(\lambda_i - \lambda_j) \int_a^b u_i u_j \, dx = 0$$

or

$$\int_a^b u_i u_j \, dx = 0 \quad : \quad \lambda_i \neq \lambda_j$$

In most applications, the integral will be zero if $i \neq j$. Functions having the above property are said to be orthogonal. It is a property of the differential

equation that eigenfunctions belonging to different eigenvalues are orthogonal. It can easily be shown that if we have the general second order linear equation:

$$u'' + P u' + Q u = \lambda u,$$

where the functions involved may be complex, the orthogonality integral is:

$$\int u_i^* u_j \, e^{\int P(x)\,dx} \, dx = A \delta_{ij}$$

Some common orthogonality integrals are:

a. $\int_0^{2\pi} \sin mx \sin nx \, dx = \delta_{mn} \pi$

b. $\int_0^{2\pi} e^{im\phi} e^{-in\phi} \, d\phi = 2\pi \delta_{mn}$

c. $\int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta = \frac{2}{2n+1} \delta_{mn}$

d. $\int_0^{\pi} P_n^m(\cos\theta) P_{n'}^m(\cos\theta) \sin\theta \, d\theta = \delta_{nn'} \frac{2\cdot(n+m)!}{(2n+1)\cdot(n-m)!}$

e. $\int_0^{\pi}\int_0^{2\pi} P_n^m e^{im\phi} P_{n'}^{m'} e^{-im'\phi} \sin\theta d\theta d\phi$

$$= \delta_{nn'} \delta_{mm'} \frac{4\pi}{2n+1} \frac{(n+m)!}{(n-m)!}$$

$$f. 2\pi \int_0^a J_0(k_m r) J_0(k_n r) r dr = -\delta_{mn} \pi a^2 (J_1(k_n a))^2$$

where $J_0(k_m a) = 0$

$$g. \int_0^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \delta_{nm} \sqrt{\pi}\, n!\, 2^n$$

Normalization: It is convenient in many applications the multiply the orthogonal functions by suitable coefficients which will make:

$$\int u_i^2 dx = 1$$

In this case we may write

$$\int u_i^* u_j dx = \delta_{ij}$$

Functions having the above property

are called orthonomal.

Expansion in Orthonormal Functions

$$f(x) = \sum_i a_i u_i(x)$$

where

$$a_j = \int u_j^* f(x)\,dx$$

If the functions have not been normalized:

$$f(x) = \sum_i a_i u_i(x)$$

$$a_j = \int u_j^* f(x)\,dx \Big/ \int |u_j|^2\,dx$$

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I Expansion in Orthogonal Functions

A. Vibrating Drum head.

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial^2 u}{\partial t^2}$$

Let

$$u = f(r\phi)\, e^{i\omega t}$$

The equation for $f(r\phi)$ has the solution

$$f(r\phi) = J_m(k_{nm} r) e^{im\phi}$$

where

$$J_m(k_{nm} a) = 0 \quad : \quad a = \text{radius of drum.}$$

The general solution is:

$$u(r\phi t) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm} J_m(k_{nm} r) e^{im\phi} e^{i\omega_{nm} t}$$

$$\int_0^a \int_0^{2\pi} u(r,\phi,0) e^{-im'\phi} J_{m'}(k_{n'm'} r)\, r\, dr\, d\phi$$

$$= \sum_{0}^{\infty} \sum_{-\infty}^{\infty} a_{nm} \int_0^a \int_0^{2\pi} J_{m'} J_m e^{i(m-m')\phi}\, r\, dr\, d\phi$$

$$= a_{n'm'} N_{n'm'}$$

where $N_{n'm'} = \int_0^a J_{m'}(k_{n'm'} r) J_{m'}(k_{n'm'} r) 2\pi r\, dr$

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I Expansion in Orthogonal Sets (cont.)

The equation:

$$\nabla^2\psi = -(k^2 + V(r))\psi$$

has special solutions of the form:

$$R_{in}(r)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$$

The function $R(r)$ depends on the function $V(r)$. The angular parts of the solution are independent of $V(r)$ and are called the Spherical Harmonics

$$Y_{nm}(\theta\phi) = P_n^{|m|}(\cos\theta)\, e^{im\phi}$$

$$\int_0^{\pi}\int_0^{2\pi} Y^*_{n'm'} Y_{nm} \sin\theta\, d\theta\, d\phi = \delta_{nn'}\delta_{mm'}\frac{4\pi}{2n+1}\frac{(n+|m|)!}{(n-|m|)!}$$

$$n \geq |m| \geq 0$$

The general solution of the equation may be written as a sum of the special solutions.

$$\psi = \sum_{i,n,m} a_{inm} R_{in}(r) P_n^{|m|}(\cos\theta)\, e^{im\phi}$$

II Illustrative Problem.



$$V = V_0 e^{i\omega t} \qquad 0 \leq \theta < \frac{\pi}{2}$$
$$V = -V_0 e^{i\omega t} \qquad \frac{\pi}{2} \leq \theta < \pi$$

In charge free space, the scalar potential obeys the equation

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0$$

Assume a periodic solution

$$\phi = f(r\,\theta\,\phi)\, e^{-i\omega t} \qquad \frac{\omega}{c} = k$$

For $r > a$

$$\nabla^2 f = -k^2 f$$

The boundary conditions on f are

$$f = V_0 \qquad 0 \leq \theta < \frac{\pi}{2}$$
$$f = -V_0 \qquad \frac{\pi}{2} \leq \theta < \pi$$

At $r \to \infty$ we wish to have only outgoing waves

$$\phi \sim e^{ik(r-ct)}$$

$$\therefore \quad f \sim e^{ikr} \qquad r \to \infty$$

$$f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{\pi}{2kr}\right)^{1/2} \left[A_{lm} J_{l+\frac{1}{2}}(kr) + B_{lm} N_{l+\frac{1}{2}}(kr) \right]$$

$$\cdot P_l^{|m|}(\cos\theta)\, e^{im\phi}$$

The boundary condition as $r \to \infty$

$f \sim e^{ikr}$

$$J_{l+\frac{1}{2}}(kr) \approx \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{l\pi}{2} - \frac{\pi}{2}\right)$$

$$N_{l+\frac{1}{2}}(kr) \approx \sqrt{\frac{2}{\pi kr}} \sin\left(kr - \frac{l\pi}{2} - \frac{\pi}{2}\right)$$

$$AJ + BN \to A\cos(kr - \delta) + B\sin(kr - \delta)$$

$$\to \alpha e^{i(kr - \delta)}$$

$$\alpha = A$$

$$i\alpha = B$$

$$A\left(J_{l+\frac{1}{2}} + iN_{l+\frac{1}{2}}\right) = A H^{(1)}_{l+\frac{1}{2}}(kr)$$

$$f = \sum_l \sum_m A_{lm} \left(\frac{\pi}{2kr}\right)^{1/2} H^{(1)}_{l+\frac{1}{2}}(kr) P_l^{|m|}(\cos\theta) e^{im\phi}$$

Boundary condition at $r = a$ $\quad f = V(\theta\phi)$

$$V(\theta\phi) = \sum_l \sum_m A_{lm} \left(\frac{\pi}{2ka}\right)^{\frac{1}{2}} H^{(1)}_{l+\frac{1}{2}}(ka)\; Y_{lm}(\theta\phi)$$

$$V(\theta\phi) = \sum_l \sum_m C_{lm}\, Y_{lm}(\theta\phi)$$

$$C_{lm} = \frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!} \int_0^{\pi} \int_0^{2\pi} V(\theta\phi)\, Y^*_{lm}(\theta\phi) \sin\theta\, d\theta\, d\phi$$

$$A_{lm} = \frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\left[\left(\frac{\pi}{2ka}\right)^{\frac{1}{2}} H^{(1)}_{l+\frac{1}{2}}(ka)\right]^{-1} \iint V Y^{*}_{lm} \sin\theta\, d\theta\, d\phi$$

In our case $V(\theta\phi) = V(\theta)$ hence $A_{lm} = 0$ for $m \neq 0$

$$V_{l0} = 2\pi \int_0^{\pi} V(\theta) P_l(\cos\theta) \sin\theta\, d\theta$$

$$V_{l0} = 2\pi V_0 \left[\int_0^{1} P_l(x)\,dx - \int_{-1}^{0} P_l(x)\,dx\right]$$

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The above integrals may be evaluated by use of the generating function.

$$1/(1-2tx+t^2)^{\frac{1}{2}} = \sum_0^{\infty} P_n(x)\, t^n$$

$$\int_0^{1} (\;)^{-\frac{1}{2}} dx - \int_{-1}^{0} (\;)^{-\frac{1}{2}} dx = \sum \left\{\int_0^{1} P_n(x)\,dx - \int_{-1}^{0} P_n(x)\,dx\right\} t^n$$

$$= \sum C_n t^n$$

$$= -\frac{1}{t}(1-2tx+t^2)^{\frac{1}{2}}\Big|_0^{1} + \frac{1}{t}(1-2tx+t^2)^{\frac{1}{2}}\Big|_{-1}^{0}$$

$$= -\frac{1}{t}(1-t) + \frac{1}{t}(1+t^2)^{\frac{1}{2}} + \frac{1}{t}(1+t^2)^{\frac{1}{2}} - \frac{1}{t}(1+t)$$

$$= \frac{2}{t}\left(\sqrt{1+t^2} - 1\right)$$

$$= t - \frac{1}{4}t^3 + \frac{1}{8}t^5 - \frac{5}{64}t^7$$

Suppose we have a source of arbitrary shape generating pure sinusoidal waves. Outside the source the wave equation is

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\psi = \sum_{ml} A_{lm} i^l \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} H^{(1)}_{l+\frac{1}{2}}(kr)\, e^{i\omega t} P_l^{(m)}(\cos\theta)\, e^{im\phi}$$

For large r, $H'_{l+\frac{1}{2}}(kr) \to \left(\frac{\pi}{2kr}\right) e^{i(kr+\delta_l)}$

$$\psi = \sum_{ml} A'_{lm} \frac{e^{i\delta_0}}{r}\, e^{ik(r-ct)}\, Y_{lm}(\theta\phi)$$

The intensity of the radiation is proportional to ψ^2

$$I = \left|\sum_{ml} A'_{lm} \frac{e^{i\delta_0}}{r}\, e^{ik(r-ct)}\, Y_{lm}(\theta\phi)\right|^2$$

$$= \frac{1}{r^2}\left|\sum_{lm} A'_{lm}\, Y_{lm}(\theta\phi)\right|^2$$

The total radiated intensity is

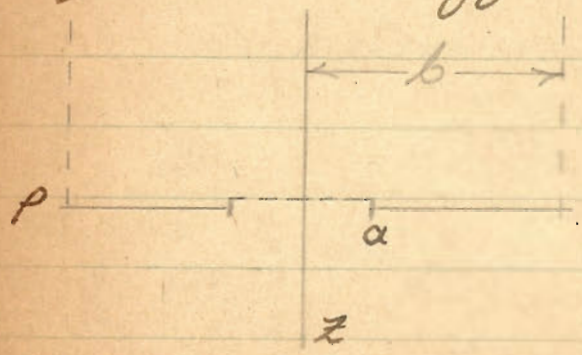
$$I_T = \int_0^{\pi}\int_0^{2\pi} I(\theta\phi)\, r^2 \sin\theta\, d\theta\, d\phi$$

$$I_T = \int_0^{\pi}\int_0^{2\pi} \sum A'^{*}_{lm} Y^{*}_{lm} \cdot \sum A'_{l'm'} Y_{l'm'} \sin\theta\, d\theta\, d\phi$$

$$= \delta_{ll'}\delta_{mm'} \frac{4\pi}{2l+1} \frac{(l+|m|)!}{(l-|m|)!} \sum |A'_{lm}|^2$$

A body radiating with the symmetry $Y_{00} = 1$ is a spherically symmetric radiator; $Y_{10} = \cos\theta$, $Y_{11} = \sin\theta\, e^{im\phi}$, $Y_{1,-1} = \sin\theta\, e^{-im\phi}$ is a dipole radiator; higher spherical harmonics are multipole radiator.

Problem: Loud speaker with a plane baffle.



$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=b} = 0$$

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = 0 \quad \rho > a$$

$$= V_0 e^{-i\omega t} \quad \rho < a$$

Outgoing waves at $z = \infty$, $\phi \sim e^{ik(r-ct)}$ $\therefore k = \frac{\omega}{c}$

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I Inhomogenious Liniar Differential equations.

$$\frac{d^2V}{dx^2} + Q(x)V = \Lambda V + S(x)$$

$$LV = \Lambda V + S(x)$$

$S(x) \equiv$ Forcing function

Consider the homogenious equation.

$$Lu_n = \lambda_n u_n$$

Suppose we know the solutions of the homogenious equation. We can expand the solutions of the inhomogenious equation in terms of the u_n.

$$V = \sum_n a_n u_n$$

$$LV = \sum_n a_n Lu_n = \sum_n a_n \lambda_n u_n$$

The inhomogenious equation becomes

$$\sum_n a_n \lambda_n u_n - \sum_n a_n \Lambda u_n = S(x)$$

$$S(x) = \sum_n s_n u_n$$

$$s_n = \int_a^b S(x)\, u_n^* \, dx$$

$$\sum_n (\lambda_n - \Lambda) a_n u_n = \sum s_n u_n$$

$$a_n = s_n / (\lambda_n - \Lambda)$$

Hence:

$$V(x) = \sum_n \frac{1}{\lambda_n - \Lambda} \left(\int_a^b S(y)\, u_n(y)\, dy \right) u_n(x)$$

This is the solution

1) Suppose $S(x)$ has the form $u_m(x)$. Only the m^{th} mode will be excited.

2) If $S(x)$ orthogonal to certain modes they will not be excited.

3) In any real problem $\Lambda \neq \lambda_n$ because there is damping. In the absence of damping the amplitude at resonance goes to infinity.

Greens Method:

$$V(x) = \int_a^b K(xy)\, S(y)\, dy$$

$K(xy)$ is called Greens function

$$K(xy) = \sum_n \frac{u_n(x)\, u_n(y)}{\lambda_n - \Lambda}$$

Substitution of $V(x)$ in the differential equation shows that $K(xy)$ obeys the equation:

$$\underline{L K - \Lambda K = \delta(x-y)}$$

Illustrative Example.

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} + f(xt)$$

Let $f(xt) = S(x)\, e^{i\omega t}$

$V(xt) = V(x)\, e^{i\omega t}$

$$\frac{d^2 V}{dx^2} = -\frac{\omega^2}{c^2} V + S(x)$$

$$\Lambda = -\frac{\omega^2}{c^2}$$

Boundary conditions $V(x,t) = 0$ for $x = 0$ and π. The solutions of the homogeneous equation is:

$$u_n = \sqrt{\frac{2}{\pi}} \sin nx$$

$$\lambda_n = -n^2 = \omega_n^2 / c^2$$

Hence:

$$V(x) = \sum_n \frac{1}{\frac{\omega^2}{c^2} - n^2} \left\{ \int_0^\pi S(y) \sin ny \, dy \right\} \sin nx$$

$$K(x,y) = \sum_n \frac{1}{\frac{\omega^2}{c^2} - n^2} (\sin ny \sin nx)$$

We can obtain $K(x,y)$ from the differential equation. This has the advantage that it sums the series.

$$\frac{d^2V}{dx^2} + \frac{\omega^2}{c^2} V = S(x)$$

$$V(x) = \frac{e^{ikx}}{2ik} \int_c^x e^{-iky} S(y) \, dy \qquad k = \frac{\omega}{c}$$

$$- \frac{e^{-ikx}}{2ik} \int_c^x e^{iky} S(y) \, dy$$

$$= \frac{1}{k} \int_c^x \sin k(x-y) \, S(y) \, dy$$

$$+ A \sin kx + B \cos kx$$

Boundary Conditions $V = 0 \quad x = 0$ and π

$$B = 0$$

$$A = -\frac{1}{k \sin k\pi} \int_0^\pi \sin k(\pi - y)\, S(y)\, dy$$

$$K(xy) = -\frac{\sin kx \sin k(\pi - y)}{k \sin k\pi} \qquad y > x$$

$$= -\frac{\sin ky \sin k(\pi - x)}{k \sin k\pi} \qquad y < x$$

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Neutron Problem: The neutron density obeys the equation.

$$\nabla^2 n + n = S$$

$n = 0$ at $r = a$

S a constant ind. of r.

$$f = \int_0^a n\, 4\pi r^2 dr$$

$$S = S\, \tfrac{4}{3}\pi a^3$$

Plot $\frac{S}{S+f}$. Show it is a straight line near $a \doteq a_0 = \pi$

I Integral Equations

$$f(x) = g(x) + \int_{-\infty}^{\infty} K(xt)\, f(t)\, dt$$

K may contain the limits of integration. This equation is linear and inhomogeneous.

I $K(xt)$ is a sum of products.

$$u(x) = 3 + \int_0^1 x t\, u(t)\, dt$$

$$= 3 + x \underbrace{\int_0^1 t\, u(t)\, dt}_{c}$$

$$u(x) = 3 + cx$$

$$c = \int_0^1 t\,(3 + ct)\, dt$$

$$c = 3\tfrac{1}{2} + \tfrac{1}{3}c$$

$$u(x) = 3 + \frac{9x}{4}$$

$$u(x) = g(x) + \int_a^b f(x) F(t) u(t) dt$$

$$u(x) = g(x) + f(x) C$$

$$C \equiv \int_a^b F(t) u(t) dt$$

A. $$C = \int_a^b F(t) [g(t) + f(x) C] dt$$

$$u(x) = 3 + \int_0^x x t\, u(t) dt$$

$$u(x) = 3 + x V(x)$$

$$V(x) = \int_0^x t\, u(t) dt$$

$$\frac{dV}{dx} = x\, u(x)$$

$$\frac{dV}{dx} = x [3 + x V]$$

This is a first order linear diff equation which we can solve.

$$u(x) = g(x) + \int_a^x f(x) F(t) u(t) dt.$$

$$u(x) = g(x) + f(x) V(x)$$

B. $$\frac{dV}{dx} = F(x) u(x)$$

$$= F(x) [g(x) + f(x) V(x)]$$

Boundary condition $V(a) = 0$

In general

$$u(x) = g(x) + \int_a^b \sum_1^n f_i(x) F_i(t)\, u(t)\, dt$$

$$= g(x) + \sum_1^n f_i(x)\, a_i$$

$$a_i = \int_a^b F_i(t)\, u(t)\, dt$$

$$a_j = \int_a^b F_j(t) \left[g(t) + \sum_{i=1}^n f_i(t)\, a_i \right] dt$$

$$u(x) = g(x) + \int_a^x \sum_1^n f_i(x) F_i(t)\, u(t)\, dt$$

$$u(x) = g(x) + \sum_1^n f_i(x)\, V_i(x)$$

$$V_i = \int_0^x F_i(t)\, u(t)\, dt$$

$$\frac{dV_i}{dx} = F_i(x)\, u(x)$$

$$\frac{dV_i}{dx} = F_i(x) \left[g(x) + \sum_i^n f_j(x)\, V_j(x) \right]$$

This method also works with non linear equations

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II Integral equations involving a displacement kernel:

$$K(x-y)$$

$$u(x) = g(x) + \int_{-\infty}^{\infty} K(x-y)\, u(y)\, dy$$

Take the Fourier transform of both sides.

$$\int_{-\infty}^{\infty} e^{i\omega x} u(x)\, dx = \int_{-\infty}^{\infty} e^{i\omega x} g(x)\, dx + \int_{-\infty}^{\infty} e^{i\omega x} dx \int_{-\infty}^{\infty} K(x-y)\, u(y)\, dy$$

$$U(\omega) = G(\omega) + I$$

Let $v = x - y$

$$I = \int_{-\infty}^{\infty} e^{iy\omega} dv \int_{-\infty}^{\infty} e^{iv\omega} K(v)\, u(y)\, dy$$

$$= \int_{-\infty}^{\infty} e^{i\omega y} u(y)\, dy \int_{-\infty}^{\infty} e^{i\omega v} K(v)\, dv$$

$$= U(\omega) L(\omega)$$

$$U(\omega) = G(\omega) + L(\omega) U(\omega)$$

$$U(\omega) = G(\omega)/(1 - L(\omega))$$

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega) e^{-i\omega x} d\omega$$

This method is based on the theorem

$$T(fg) = \int F(\omega - \omega') G(\omega') d\omega'$$

If we have

$$u(x) = g(x) + \int_{-\infty}^{x} K(x-y) u(y) dy$$

$$v = x - y$$

$$\int_{+\infty}^{0} e^{i\omega v} K(v) dv \int_{-\infty}^{\infty} e^{i\omega y} u(y) dy$$

where we have defined

$$K(v) = K(v) \quad v > 0$$
$$K(v) = 0 \quad v < 0$$

III Orthogonality of Functions involved in integral equations

Suppose the kernel is symmetrical in x and y

$$K(xy) = K(yx)$$

Consider the equation

$$\alpha V(x) = \int K(xy) V(y)\, dy$$

$$\alpha V(x) = A V(x)$$

A is a linear operator

$$A(u+v) = Au + Av$$

$$\int K(xy)(u+v)\, dy = \int Ku\, dy + \int Kv\, dy$$

The equation $\alpha V = AV$ has solutions for certain values of α.

$$\alpha_n V_n = A V_n$$

$V_m \parallel \quad \alpha_n V_n = \int K(xy) V_n\, dy$

$V_n \parallel \quad \alpha_m V_m = \int K(xy) V_m\, dy$

$$(\alpha_n - \alpha_m) \int V_n V_m\, dx = \int\int dy\, dx - \int\int dy\, dx$$

$$= \{ \iint K(xy) V_n(x) V_m(y)\, dx\, dy$$

$$- \iint K(xy) V_m(x) V_n(y)\, dx\, dy \}$$

If K symmetrical the above term $= 0$

$$\int V_n(x) V_m(x)\, dx = \delta_{nm} : \text{constant}$$

Problems:

1. $u(x) = e^{-x} + \int_0^x (t-x)^2 u(t)\, dt$

2. $f(x) = \frac{1}{x^3} + 2 \int_0^1 f\left(\frac{x}{y}\right)\left(\frac{1}{y} - 1\right) \frac{dy}{y}$

3. $g(x) = \int_0^x \frac{f(t)}{(t-x)^s}\, dt : 1 > s > 0$
given $g(x)$ find $f(t)$

IV Approximate Methods of Solution

1. Iteration

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$$A V_n = \alpha_n V_n$$

We assume:

1. A is linear
2. A is self adjoint (Hermitian)

i.e. $\int \psi (A\psi)\, dx = \int (A\psi)\psi\, dx$
for all ψ and ψ.

The condition that an integral operator be self-adjoint is:

$$A = \int K(xy) \ldots dy$$

$$\int \psi \left(\int K(xy)\, \psi\, dy \right) dx$$

$$= \int \left(\int K(xy)\, \psi\, dy \right) \psi\, dx$$

These two integrals are equal if $K(xy) = K(yx)$ i.e. $K(xy)$ symmetric.

Definition $\tilde{A}$ = adjoint to A if

$$\int \psi (A\psi)\, dx = \int (\tilde{A}\psi)\psi\, dx \text{ for all } \psi, \psi$$

Suppose we know solutions of the equation $A v_n = \alpha_n v_n$. We can find solutions of the equation:

$$A v - \alpha v = f(x)$$

$$v = \sum b_n v_n$$

$$\sum b_n A v_n - \alpha \sum b_n v_n = f(x)$$

$$\sum b_n (\alpha_n - \alpha) v_n(x) = f(x)$$

Expand: $f(x) = \sum_n c_n v_n(x)$

$$c_n = \int v_n(x) f(x)\, dx$$

$$\sum b_n (\alpha_n - \alpha) v_n(x) = \sum c_n v_n(x)$$

$$b_n = \frac{c_n}{\alpha_n - \alpha}$$

Hence.

$$v = \sum \frac{c_n v_n(x)}{\alpha_n - \alpha}$$

$$V = \sum_n \frac{V_n(x)}{a_n - a} \int V_n(y) f(y)\, dy$$

$$= \int f(y) \left\{ \sum \frac{V_n(x) V_n(y)}{a_n - a} \right\} dy$$

Consider the equation:

$$Ag(x) = f(x) = a$$

It is clear from the above that:

$$g(x) = \int f(y) \left(\sum \frac{V_n V_n}{a_n} \right) dy$$

$$f(x) = \int K(xy)\, g(y)\, dy$$

We define $K_{-1}(xy) = \sum \frac{V_n(x) V_n(y)}{a_n}$ to be A^{-1}

$$g(x) = A^{-1} f(x)$$

$$A\, g(x) = A A^{-1} f(x) = f(x)$$

We can expand $K(xy)$ in terms of the $V_n(x)$

$$K(xy) = \sum_n k_n(x)\, v_n(y) = A$$

$$\int K(xy)\, v_n(y)\, dy = \alpha_n v_n(x)$$

$$\int \left(\sum_m k_m(x)\, v_m(y)\right) v_n(y)\, dy = \alpha_n v_n(x)$$

$$k_n(x) = \alpha_n v_n(x)$$

$$K(xy) = A = \sum_n \alpha_n v_n(x)\, v_n(y)$$

$$K(xy) = A^{-1} = \sum \frac{1}{\alpha_n} v_n(x)\, v_n(y)$$

If we have any equation differential or integral of the form:

$$L u_n = \lambda_n u_n$$

we can find an integral operator equivalent to L. (in principle:

$$L = \frac{d^2}{dx^2} + Q(x)$$

$$L(xy) = \sum \lambda_n u_n(x)\, u_n(y)$$

$$\int L(xy) F(y)\, dy = \frac{d^2 F}{dx^2} + Q(x) F$$

In a real problem the Σ diverges, but by taking appropriate limits we get

$$L(xy) = \underbrace{\delta''(x-y)}_{\frac{d^2}{dx^2}} + \delta(x-y)\,Q(x)$$

This leads to the original equation so nothing was gained.

Consider:

$$A V_n = a_n V_n$$

If B is the inverse operator

$$B A V_n = V_n = B(a_n V_n)$$

$$= a_n B V_n = V_n$$

Hence:

$$B V_n = \frac{1}{a_n} V_n$$

Suppose O an operator such that

$$O V_n = a_n^2 V_n$$

$$O(xy) = \Sigma a_n^2 V_n(x) V_n(y)$$

Consider $\int K(xz)\, K(zy)\, dz$

$$= \int \sum \alpha_n V_n(x) V_n(y) \cdot \sum \alpha_m V_m(z) V(y)\, dz$$

$$= \sum \alpha_n \alpha_m V_n(x) V_m(y) \int V_n(z) V_m(z)\, dz$$

$$= \sum \alpha_n^2 V_n(x) V_m(y) = O(xy)$$

$$O(xy) \equiv A^2$$

In general we can calculate:

$$A(A V_n) = A \alpha_n V_n = \alpha_n A V_n$$

$$= \alpha_n^2 V_n$$

In the most general case we have:

$f(A)$ has the same eigenfunctions as A and the eginvalues $f(\alpha_n)$ where $A V_n = \alpha_n V_n$

$$f(A) V_n = f(\alpha_n) V_n$$

$$f(A)\ldots \equiv \int \sum f(\alpha_n) V_n(x) V_n(y) \ldots dy$$

The operators, A_i, obey all the rules of algebra except the commutative law

$$AB \neq BA$$

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$$f(A) = I f(0) + A f'(0) + \frac{A^2}{2} f''(0)$$

where: $I \equiv \int \delta(x-y) \ldots dy$

Proof:

$$f(A) V_n(x) = \int \delta(x-y) V_n(y) dy$$

$$+ \int \sum_i a_i V_i(x) V_i(y) V_n(y) dy f'(0) + \ldots$$

$$= f(0) V_n(x) + f'(0) a_n V_n(x) + \ldots$$

$$= V_n(x) f(a_n)$$

Expansion of operators in terms of matrix elements:

a. $B f_n = \sum_k c_k f_k$

$$c_m = \int f_m B f_n \, d\tau \equiv B_{mn}$$

Hence

$$B f_n = \sum_m B_{mn} f_m$$

b. $B(A f_n) = B \sum_m A_{mn} f_m$

$$= \sum_m A_{mn} B f_m = \sum_m A_{mn} \sum_k B_{km} f_k$$

$$= \sum_k \left(\sum_m B_{km} A_{mn} \right) f_k$$

$$(BA) f_n = \sum_k (BA)_{kn} f_k$$

Hence $\sum_m B_{km} A_{mn} = (BA)_{kn}$

Suppose we have solved the problem

$A V_n = a_n V_n$ and have expanded a function $f(x)$

in terms of the eigenvalues $V_n(x)$

$$f(x) = \sum_n f_n V_n(x)$$

We wish to find a solution of the equation

$$g(x) = \int M(xy) f(y)\, dy$$

$$= \int M(xy) \sum_n f_n V_n(y)\, dy$$

Expand $g(x)$

$$\sum_m g_m V_m(x) = \sum_n f_n \int M(xy) V_n(y)\, dy$$

$$g_m = \sum_n M_{mn} f_n$$

where $M_{mn} = \int\int V_m(x) M(xy) V_n(y)\, dy\, dx$

and

$$M(xy) = \sum_{nm} M_{nm} V_n(x) V_m(y)$$

The following table gives operators and their matrix element expansions:

$g(x) = \sum_n g_n V_n(x)$. $f(x) = \sum_m f_m V_m(x)$

$g(x) = \int A(xy) f(y)\, dy$	$g_n = \sum_m A_{nm} f_m$
A	A_{nm}
B	B_{nm}
$A + B$	$A_{nm} + B_{nm}$
AB	$(AB)_{nm}$
$= \int A(xz) B(zy)\, dz$	$= \sum_k A_{nk} B_{km}$
$\delta(x-y)$	δ_{nm}

where $A_{nm} = \iint V_n(x) A V_m(y)\, dy\, dx$

The following operator relations are <u>not</u> true:

$$A V_n = a_n V_n \qquad B W_n = \beta_n W_n$$

$$AB \neq BA$$

$$A + B \neq a_n + \beta_n$$

$$AB \neq a_n \beta_n$$

I Variational Methods

Works only for a symmetrical kernel.

Consider:

$$\int K(xy) V_n(y)\, dy = a_n V_n(x)$$

Let

$$I = \iint f(x)^* K(x,y) f(y)\, dx\, dy$$

$$f(x) = \sum_n c_n v_n(x)$$

$$I = \iint \sum_{mn} c_m v_m(x) K(x,y) c_n v_n(y)\, dx\, dy$$

$$= \int \sum_{mn} c_m v_m(x) c_n \alpha_n v_n(x)\, dx$$

$$= \sum_n \alpha_n (c_n)^2$$

Suppose there is an extreme eigenvalue. We shall assume a smallest α_0

$$I = \alpha_0 \sum_n c_n^2 + \sum_n (\alpha_n - \alpha_0) c_n^2$$

$$I \geq \alpha_0 \sum_n (c_n)^2$$

$$\sum_n c_n^2 = \int |f(x)|^2 dx$$

$$I \geq \alpha_0 \int |f(x)|^2 dx$$

$$\alpha_0 \leq \iint f(x) K(x,y) f(y)\, dx\, dy \,/\, \int |f(x)|^2 dx = \alpha_f$$

The equality holds when $f(x) = V_0(x)$

$$\alpha_0 = \alpha_{V_0}$$

For integral operators the method converges very rapidly.

$$\alpha_{(V_0+\delta)} = \alpha_0 + O(\delta^2)$$

The method is applicable to differential equations

$$\frac{d^2 u_n}{dx^2} + Q(x) u_n = \lambda_n u_n$$

$$I = \int f(x) \left(\frac{d^2}{dx^2} + Q(x) \right) f(x) \, dx$$

$$\lambda_0 \leq \int \left[Q(x) f^2(x) - \left(\frac{df}{dx} \right)^2 \right] dx \Big/ \int f^2(x) \, dx$$

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Integral equations usually have no constants of integration

Application of Variational Method.

Consider:

$$\frac{d^2u}{dx^2} = -\alpha u$$

where $u = 0$ at $x = \pm 1$. The exact solution is $u = \cos\sqrt{\alpha}\,x$, $\sqrt{\alpha} = \pi/2$. The lowest eigenvalue is $\alpha = \pi^2/4 = 2.4674$

$$\lambda_0 \doteq \int [Q(x) f^2(x) - (\frac{df}{dx})^2]\,dx \Big/ \int f^2(x)\,dx$$

$$= \int_{-1}^{1} (\frac{df}{dx})^2 dx \Big/ \int_{-1}^{1} f^2\,dx$$

First try $f(x) = 1 - x^2$. It is better to use a solution containing parameters.

$$f = 1 - ax^2 + (a-1)x^4$$

$$f' = -2ax + 4(a-1)x^3$$

$$\int_{-1}^{1} f'^2\,dx = \int_{-1}^{1} 4a^2x^2 - 16a(a-1)x^4 + 16(a-1)^2x^6$$

$$= \frac{8}{3}a^2 - \frac{32a(a-1)}{5} + \frac{32(a-1)^2}{7}$$

$$= 2\left[.419a^2 - 1.371a + 2.286\right]$$

$$\int_{-1}^{1}\left[1 - ax^2 + (a-1)x^4\right]^2 dx$$

$$= 2\left[1 - \frac{2a}{3} + \frac{a^2}{5} + \frac{2(a-1)}{5} - \frac{2a(a-1)}{7} + \frac{(a-1)^2}{9}\right]$$

$$= 2\left[.7111 - .2023a - .0254a^2\right]$$

$$\lambda \leq (.419a^2 - 1.371a + 2.286)/(.7111 - .2023a - .0254a^2)$$

If $a = 1$ $\lambda_1 = \frac{1.334}{.533} = 2.5$ which is fairly accurate.

a	N	D	N/D
1.000	1.334	.5333	2.500
2.000	1.221	.4063	3.000
.500			2.77
1.300			2.47?
1.100			2.481
1.200			2.469
1.250			2.468

For $a = 1.25$

$\lambda = 2.468$

The method converges even faster for integral equations.

Problem: $\alpha \cdot u(x) = \int_{-1}^{1} e^{-|x-y|} u(y)\,dy$

Find the lowest eigenvalue.

Try $u(y) = 1$, then $u(y) = 1 - \delta y^2$

Best value of δ and α ?

Solve exactly (optional).

For unsymmetrical kernels there is no general method of solution.

Some special cases:

$$\alpha\, v(x) = \int K(x,y)\, P(y)\, dy\, v(y)$$

$K(x,y)$ symmetric

$$\alpha \sqrt{P(x)}\, v(x) = \int \underbrace{\sqrt{P(x)}\, K(x,y) \sqrt{P(y)}}_{K'(x,y)}\, v(y) \sqrt{P(y)}\, dy$$

This gives a new kernel K' for $V(x)\sqrt{P(x)}$ which is symmetric and the variational method may be applied:

$$\lambda = \int\int f(x) K(xy) f(y) P(x)\,dx\,P(y)\,dy \Big/ \int f^2(x) P(x)\,dx$$

In the general case, nothing is known.

Suppose

$$\alpha f(x) = \int M(xy) f(y)\,dy$$

$M(xy)$ not symmetrical

It is known:

1) only special values of α_n and functions f_n give solutions

2) f_n's are not orthogonal

Consider adjoint equation with adjoint kernel

$$\beta P(x) = \int M(yx) P(y)\,dy$$

This also has special values and functions β_n and P_n for

which a solution exists.

$$\beta_n = \alpha_n$$

$$f_n \neq \rho_n$$

and $\int f_n(x)\, \rho_m(x)\, dx = \delta_{mn}$

α_n is complex except for smallest

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Linear Algebraic Equations

$$y_1 = \alpha_{11} x_1 + \alpha_{12} x_2 + \dots + \alpha_{1N} x_N$$

$$y_2 = \alpha_{21} x_1 + \alpha_{22} x_2 + \dots + \alpha_{2N} x_N$$

or

$$y_i = \sum_{j=1}^{N} \alpha_{ij} x_j$$

or

$$y = \alpha x$$

Where:

α is a matrix which characterizes the above systems of equations:

$$\alpha = \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \dots \\ \alpha_{21} & \alpha_{22} & \dots & \\ \alpha_{31} & \dots & & \\ \dots & & & \end{matrix}$$

x and y are one column matrices or vectors.

Matrix Algebra

$$y = \alpha x \qquad y_i = \sum_j \alpha_{ij} x_j$$

$$z = \beta y \qquad z_i = \sum_j \beta_{ij} y_j$$

$$z = \gamma x \qquad z_i = \sum_j \gamma_{ij} x_j$$

Find γ_{ij}

$$z_i = \sum_k \sum_j \beta_{ik} \alpha_{kj} x_j$$

$$\gamma_{ij} = \sum_k \beta_{ik} \alpha_{kj} = (\beta\alpha)_{ij}$$

$$\gamma = \beta\alpha$$

This is the rule for multiplying matrices. Graphically:

	1	2	3
1	A	B	C
2			
3			

	1'	2'	3'
1'	A'		
2'	B'		
3'	C'		

	1	2
i	AA'+BB'+CC'	---
= 2	---	
3		

Association Law

$$[\delta(\beta\alpha)]_{ij} = \sum_l \delta_{il}(\beta\alpha)_{lj}$$

$$= \sum_l \delta_{il} \sum_k \beta_{lk}\alpha_{kj} = \sum_k \left(\sum_l \delta_{il}\beta_{lk}\right)\alpha_{kj}$$

$$= [(\delta\beta)\alpha]_{ij}$$

$$\delta(\beta\alpha) = (\delta\beta)\alpha \equiv \delta\beta\alpha$$

Rotations can be represented by matrices.

$$X' = R_x X$$

$$R_x = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{Vmatrix}$$ Rotation of 90° around x-axis

$$R_y = \begin{Vmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}$$ Rotation of 90° around y axis

Problem $R_y R_x =$

and $R_x R_y =$

Show that if R_Δ represents rotation by 120° about the body diagonal of a cube, then

$$R_\Delta R_y R_x = R_x R_y$$

$$R_\Delta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

Example $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Show $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$

$\sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$

$\sigma_y \sigma_z = -\sigma_z \sigma_y = i\sigma_x$

$\sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y$

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There is a close connection between the theory of integral equations we have just studied and matrices

$$y_i = \sum_j a_{ij} x_j$$

$$\rightarrow G(s) = \int K(st) F(t) dt$$

$$(B\alpha)_{ij} = \sum_k \beta_{ik} \alpha_{kj}$$

$$\rightarrow P(st) = \int M(sw) K(wt) dw$$

The Unit Matrix

$$I \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \cdots & & & \cdots & \end{pmatrix}$$

The elements of the unit matrix are δ_{ij}

$$I\alpha = \alpha \qquad \alpha_{ij} = \sum_k \left\{ \begin{matrix} \delta_{ik} \\ I_{ik} \end{matrix} \right\} \alpha_{kj}$$
$$\alpha I = \alpha$$

A diagonal matrix is one having elements only on its major diagonal.

A Reciprocal matrix is defined by:

$$\beta\alpha = I$$

$\beta \equiv$ reciprocal of $\alpha \equiv \alpha^{-1}$

$$\alpha^{-1}\alpha = I$$

$$\alpha(\alpha^{-1}\alpha) = (\alpha\alpha^{-1})\alpha = \alpha I = I\alpha$$

$$\alpha\alpha^{-1} = I$$

Rules for operating with matrices:

$$(\alpha+\beta)X = \alpha X + \beta X$$

$$(\alpha+\beta) = (\beta+\alpha)$$

$$\gamma(\alpha+\beta) = \gamma\alpha + \gamma\beta$$

$$(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$$

$$c\,\alpha_{ij} = (c\alpha)_{ij}$$

$$F(\alpha)G(\alpha) = G(\alpha)F(\alpha)$$

$$\therefore \alpha\beta \neq \beta\alpha$$

Illustrative examples

a. $\beta = (I + 3\alpha + 2\alpha^2)$

where

$$\alpha = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 14 & 6 \end{pmatrix}$$

b. $\beta = (I + 2\alpha)(I + \alpha)$

$= I + 2\alpha + \alpha + 2\alpha^2$

$= (I + \alpha)(I + 2\alpha)$

Problem: Show

$$\sigma_x \sigma_y - \sigma_y \sigma_x = 2i\sigma_z$$

$$(\sigma_x + i\sigma_y) = (\sigma_z - 2)(\sigma_x + i\sigma_y)$$

In general, if we are given the structure of a closed set of matrices, we can not find the individual matrices.

Suppose we are given a set of matrices which obey certain combination rules. We can construct another set of matrices which will obey the same rules as follows. Let S be any matrix with a reciprocal, and let $\alpha, \beta, \gamma \ldots$ be given. The matrices

$$\alpha' = S\alpha S^{-1}$$
$$\beta' = S\beta S^{-1}$$
$$\gamma' = S\gamma S^{-1}$$

will obey the same rules as $\alpha, \beta, \gamma, \ldots$

$$S(\alpha+\beta)S^{-1} = S(\alpha S^{-1} + \beta S^{-1})$$
$$= S\alpha S^{-1} + S\beta S^{-1}$$

Suppose $\alpha\beta = \delta$

$$S\alpha\beta S^{-1} = S\delta S^{-1}$$
$$S\alpha S^{-1} S\beta S^{-1} = \delta'$$
$$\alpha'\beta' = \delta'$$

Sets of matrices which have the same relation between their sums and products are called "equivalent" matrices.

A "representation" of a set of matrices is a typical group of matrices which have the properties of the set.

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Invariants of a Matrix

1. Trace or Spur of a Matrix

$$Tr(\alpha) \equiv \sum_i a_{ii}$$

$$Tr(\alpha+\beta) = Tr(\alpha) + Tr(\beta)$$

$$Tr(\alpha\beta) = \sum_i \sum_k a_{ik} b_{ki} = Tr(\beta\alpha)$$

$$Tr(\alpha\beta\gamma\delta) = Tr(\delta\alpha\beta\gamma) = Tr(\gamma\delta\alpha\beta)$$

$$Tr(I) = N \text{ order of matrix.}$$

2. Determinant of a Matrix

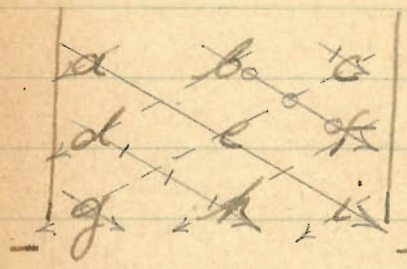
$D(a)$ = Σ all products of elements of a one from each row and column (with appropriate signs)

2 × 2 Det



$$= ab - cd$$

3 × 3 Det



$$= aei + bfg + cdh - ceg - fha - ibd$$

Rules:

1. Interchanging two rows or columns of a determinant changes the sign.
2. We can add to any row (column) any multiple of any

row (column) without changing the matrix.

3. Multiplication by a factor multiplies any row (column) by the factor

$$\begin{vmatrix} 2 & 1 & 4 \\ 7 & 1 & 7 \\ 2 & 6 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 6 \\ 7 & 1 & 14 \\ 2 & 6 & 5 \end{vmatrix}$$

$$= \frac{1}{6} \begin{vmatrix} 2 & 6 & 4 \\ 7 & 6 & 7 \\ 2 & 36 & 3 \end{vmatrix}$$

4 Reducing the order of a determinant

A. Expansion in minors.

The minor, A_{ij}, of a determinant is the determinant with the i^{th} row and j^{th} column struck out times the factor $(-1)^{i+j}$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & & & \\ a_{N1} & a_{N2} & \dots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{ij} \cdot A_{ij}$$

$$= a_{i1} A_{i1} + a_{i2} A_{i2} \dots a_{in} A_{in}$$

$$\begin{vmatrix} 2 & 1 & 4 \\ 7 & 1 & 7 \\ 2 & 6 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 7 \\ 6 & 3 \end{vmatrix} - 1 \begin{vmatrix} 7 & 7 \\ 2 & 3 \end{vmatrix} + 4 \begin{vmatrix} 7 & 1 \\ 2 & 6 \end{vmatrix}$$

B. Large determinants are best reduced as follows:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1i} & \dots & a_{1N} \\ a_{21} & a_{22} & & & & \\ \dots & & & & & \\ a_{j1} & & & a_{ji} & & \\ \dots & & & & & \\ a_{N1} & & & & & a_{NN} \end{vmatrix}$$

$$= \frac{1}{(a_{11})^{N-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{vmatrix} & \cdots & \cdots & \begin{vmatrix} a_{11} & a_{1N} \\ a_{21} & a_{2N} \end{vmatrix} \\ \cdots & & & \\ \cdots & \begin{vmatrix} a_{11} & a_{1i} \\ a_{j1} & a_{ji} \end{vmatrix} & & \\ \cdots & & & \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{N1} & a_{N2} \end{vmatrix} & \cdots & \cdots & \begin{vmatrix} a_{11} & a_{1N} \\ a_{N1} & a_{NN} \end{vmatrix} \end{vmatrix}$$

To simplify calculations make $a_{11} = 1$ by manipulating rows & columns.

$$\mathrm{Det}(\alpha+\beta) \neq D(\alpha) + D(\beta)$$
$$D(\alpha\beta) = D(\alpha)\cdot D(\beta) = D(\beta\alpha)$$
$$D(s\alpha s^{-1}) = D(\alpha)$$

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Definition: Singular Matrix

$$D(\alpha) = 0$$

If α singular its reciprocal does

not exist.

$$\beta\alpha = 1$$

$$D(B\alpha) = D(\beta)D(\alpha) = D(1) = 1$$

$$\therefore \quad D(\beta) = \infty$$

If $D(\alpha) \cdot D(\beta) = 0$ either $D(\alpha)$ or $D(\beta) = 0$.

If we know α is not singular

$$\alpha\beta = 0$$

$$\alpha^{-1}\alpha\beta = \alpha^{-1} \cdot 0$$

$$\therefore \quad \beta = 0$$

Determinant of the reciprocal matrix

$$\alpha\beta = 1$$

$$(\alpha\beta)_{ki} = 1_{ki} = \delta_{ki}$$

$$\sum_j \alpha_{kj} \beta_{ji} = \delta_{ki}$$

Comparing this with

$$\sum_j \alpha_{ij} A_{ij} = D(\alpha)$$

we see $B_{ji} = A_{ij} / D(\alpha)$

Suppose we have a system of equations

$$\alpha X = y$$

which we want to solve for X $\quad X = \alpha^{-1} y$

$$x_i = \sum_j (\alpha^{-1})_{ij} y_j$$

$$= \sum_j \frac{A_{ji}}{D(\alpha)} y_j$$

$$= \frac{D(\alpha) \text{ with } i^{th} \text{ column replaced by } y}{D(\alpha)}$$

If there is a system of equations

$$\alpha u = 0$$

and not all $u = 0$, the $D(\alpha) = 0$

Definition: Consider a matrix

$$\alpha \to \alpha_{ij}$$

a The Transpose (Adjoint) $\tilde{\alpha} = \alpha_{ji}$.

b) Complex conjugate $\alpha^* = \alpha_{ij}^*$

c) Hermetian adjoint $\alpha^\dagger = \alpha_{ji}^*$

If $\tilde{\alpha} = \alpha$ the matrix is symmetrical

If $\alpha^\dagger = \alpha^{-1}$ " " " unitary

If $\tilde{\alpha} = \alpha^{-1}$ " " " orthogonal

If $\alpha^\dagger = \alpha$ " " " Hermitian

Systems of Equations

$$\alpha X = \lambda X$$

$$\sum_j a_{ij} x_j = \lambda x_i$$

$$\sum_j (a_{ij} x_j - \lambda \delta_{ij} x_j) = 0$$

$$(\alpha - \lambda 1) X = 0$$

Not all X are 0 hence

$$D(\alpha - \lambda 1) = 0$$

The λ's are the eigenvalues of the matrix α

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22}-\lambda & \cdots & \\ a_{31} & \cdots & a_{33}-\lambda & \\ \cdots & & & \cdots \end{vmatrix} = 0$$

Esp to each eigenvalue $\lambda^{(n)}$ there is an eigenvector $X_i^{(n)}$ such that

$$\alpha X^{(n)} = \lambda^{(n)} X^n$$

Eigenvalues of equivalent matrices are equal, i.e. they are invariant to the transformation $S \alpha S^{-1}$

$$\alpha X = \lambda X$$
$$\alpha' = S \alpha S^{-1}$$
$$\alpha' y = \lambda' y$$
$$S \alpha S^{-1} y = \lambda' y$$
$$\alpha (S^{-1} y) = \lambda' (S^{-1} y)$$
$$\alpha \underbrace{X} = \lambda \underbrace{X}$$

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Any real symmetrical matrix can be made equivalent to a diagonal matrix.

- or -

There exist some S such that $S\alpha S^{-1}$ is diagonal if α is a real symmetrical matrix.

I Diagonal Matrices

$$\alpha^D Z = \lambda Z$$

$$\sum_j \alpha^D_{ij} Z_j = \lambda Z_j$$

$$\alpha^D_{ii} Z_i = \lambda Z_i$$

$$\alpha^D_{ii} = \lambda$$

Hence the eigenvalues of a diagonal matrix are its diagonal elements and the eigenfunctions are the components of the vector on which it operates

$$\lambda^{(n)} = \alpha^D_{nn} \qquad Z^{(n)}_i = Z^{(n)}_n \delta_{in}$$

Let S be the matrix which diagonalizes α where $\alpha X = \lambda X$

$$S\alpha S^{-1} = \alpha^D$$
$$S^{-1}\alpha^D S X = \lambda X$$
$$\alpha^D \underbrace{(SX)} = \lambda(SX)$$
$$\alpha^D Z = \lambda Z$$

We know the solutions of this equation and X is related to Z by

$$X = S^{-1} Z$$

Now suppose we know the solutions of the equation $\alpha X = \lambda X$ are $\lambda^{(n)} + X_i^{(n)}$

$$\sum_i X_i^{(m)} \Big\| \sum_j a_{ij} X_j^{(n)} = \lambda^{(n)} X_i^{(n)}$$

$$\sum_j X_j^{(m)} \Big\| \sum_i X_i^{(m)} a_{ji} = \lambda^{(m)} X_j^{(m)}$$

$$\sum_{ij} X_i^{(m)} a_{ij} X_j^{(n)} = \lambda^{(n)} \sum_i X_i^{(m)} X_i^{(n)}$$

$$\sum_{ji} X_i^{(m)} a_{ji} X_j^{(n)} = \lambda^{(m)} \sum_j X_j^{(n)} X_j^{(m)}$$

Since $a_{ij} = a_{ji}$

$$(\lambda^{(n)} - \lambda^{(m)}) \sum_i X_i^{(n)} X_i^{(m)} = 0$$

$\therefore$

$$\sum_i X_i^{(n)} X_i^{(m)} = \delta_{nm}$$

Let us consider the $N \times N$ matrix of all the solutions $X_i^{(n)}$ of the equation $\alpha X = \lambda X$. Call this matrix

$$T \sim T_{ki} = X_i^{(k)}$$

$$\sum_i T_{ni} T_{mi} = \delta_{nm}$$

$$\sum_i T_{ni} \tilde{T}_{im} = \delta_{nm}$$

$$T \tilde{T} = I$$

$$\tilde{T} = T^{-1}$$ orthogonal matrix

Let $\alpha' = T \alpha T^{-1}$

$= T \alpha \tilde{T}$

$$(\alpha')_{ij} = \sum_k \sum_l T_{ik} a_{kl} \tilde{T}_{lj}$$

$$= \sum_k \sum_\ell X_k^{(i)} \alpha_{k\ell} X_\ell^{(j)}$$

$$= \sum_k X_k^{(i)} \underbrace{\sum_\ell \alpha_{k\ell} X_\ell^{(j)}}_{\lambda^{(j)} X_k^{(j)}}$$

$$(\alpha')_{ij} = \sum_k X_k^{(i)} X_k^{(j)} \lambda^{(j)}$$

$$= \lambda^{(j)} \delta_{ij}$$

$$\alpha' = \alpha^D = T \alpha T^{-1}$$

where: $T_{ik} = X_i^{(k)}$

$$\alpha^D Z^{(n)} = \lambda^{(n)} Z^{(n)}$$

$$X^{(n)} = T^{-1} Z^{(n)}$$

T is the matrix which diagonalizes α.

$$T_{ni} = (\tilde{T})_{in} = (T^{-1})_{in} = X_i^{(n)}$$

The eigenvalues are the only invariants of a symmetrical matrix.

$$\alpha^P x = \lambda x$$

$$(\alpha - \lambda 1) x = 0$$

$$D(\alpha - \lambda 1) = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) \ldots$$

$$D(\alpha) = D(\alpha^D) = \prod_i \lambda^i$$

$$Tr(\alpha) = Tr(\alpha^D) = \sum_i \lambda^i$$

Example:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$0 = (1-\lambda)(\lambda^2 + 1)$$

$$\lambda^{(1)} = 1$$
$$\lambda^{(2)} = i$$
$$\lambda^{(3)} = -i$$

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$$R_x \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ -X_3 \\ X_2 \end{pmatrix} = \begin{pmatrix} \lambda X_1 \\ \lambda X_2 \\ \lambda X_3 \end{pmatrix}$$

Solutions

(1) $\lambda^{(1)} = 1$	(2) $\lambda^{(2)} = i$	(3) $\lambda^{(3)} = -i$
$X_1^{(1)} = 1$	$X_1^{(2)} = 0$	$X_1^{(3)} = 0$
$X_2^{(1)} = 0$	$X_2^{(2)} = 1$	$X_2^{(3)} = 1$
$X_3^{(1)} = 0$	$X_3^{(2)} = -i$	$X_3^{(3)} = i$

II. Method of finding solutions (eigenvectors) when λ's are known.

$$\alpha X = \lambda X.$$
$$(\alpha - \lambda 1) X = 0$$

Let $(\alpha - \lambda 1) = \beta$

$$\beta X = 0$$
$$\sum_j \beta_{ij} X_j = 0$$

For non-trivial solutions $D(\beta) = 0$

$$D(\beta) = \sum_j \beta_{ij} B_{kj} = 0$$

where B_{kj} is the kj minor of β. Comparing this with the equation $\sum_j \beta_{ij} X_j = 0$ we see that

$$X_j^{(\lambda)} = B_{kj}^{(\lambda)}$$

The different values of k will give trivial or the same solutions times a factor.

III. Theory of Small Vibrations

x = displacement from equilibrium

$$V = V(x_i) \qquad i = 1 \ldots 3N$$

Expand V about the equilibrium position

$$V = 0 + 0 + \frac{1}{2} \sum_{ij} k_{ij} x_i x_j + \ldots$$

$$F_i = -\frac{\partial V}{\partial x_i} = -\sum_j k_{ij} x_j$$

we have assumed $k_{ij} = k_{ji}$

$$m_i \ddot{x}_i = -\sum_j k_{ij} x_j$$

$$M = m_i \delta_{ij} \quad ; \quad K = k_{ij}$$

$$M\ddot{x} = -Kx$$

Let

$$x = \mu e^{i\omega t}$$
$$\ddot{x} = -\omega^2 \mu e^{i\omega t}$$

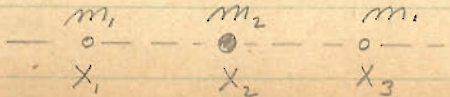
$$\omega^2 M\mu = K\mu$$
$$(K - \omega^2 M)\mu = 0$$

To get ω's

$$D(K - \omega^2 M) = 0$$

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I The linear vibrations of a CO_2 molecule:



A knowledge of the ω's enables one to compute the fundamental modes of vibration.

Let $\alpha = T^{-1} \alpha_p T$

where α is a symmetrical matrix $T_{ki} = x_i^{(k)}$

$$\alpha_{ij} = \sum_k \sum_l (T^{-1})_{ik} \alpha_{p_{kl}} T_{lj}$$

$$\alpha_{p_{kl}} = \lambda^{(k)} \delta_{kl}$$

$$\alpha_{ij} = \sum_k T_{ki} \lambda^{(k)} T_{kj}$$

$$\alpha_{ij} = \sum_k \lambda^{(k)} x_i^{(k)} x_j^{(k)}$$

We see that the matrix elements of an operator can be expanded in terms of the eigenvalues and eigenvectors of the operator.

For small displacements, we assume the force proportional to the displacement.

1) $m_2 \ddot{X}_2 = -a(X_2 - X_3) - a(X_2 - X_1)$

$k_{22} = 2a$

$k_{21} = -a$

$k_{23} = -a$

$m_1 \ddot{X}_1 = -a(X_1 - X_2)$

$k_{11} = a$

$k_{12} = -a$

$m_1 \ddot{X}_3 = -a(X_3 - X_2)$

$k_{33} = a$

$k_{32} = -a$

$$\text{Det.}\begin{pmatrix} a - m_1\omega^2 & -a & 0 \\ -a & 2a - m_2\omega^2 & -a \\ 0 & -a & a - m_1\omega^2 \end{pmatrix} = 0$$

Solving this determinant we get:

1) $\omega^2 = \dfrac{a}{m_1}$

2) $\omega^2 = 0$

3) $\omega^2 = a\,\dfrac{2m_1 + m_2}{m_1 m_2}$

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A knowledge of the ω's enables one to compute the fundamental modes of vibration.

Let $\alpha = T^{-1} \alpha_p T$

where α is a symmetrical matrix $T_{ki} = x_i^{(k)}$

$$\alpha_{ij} = \sum_k \sum_l (T^{-1})_{ik} \alpha_{pkl} T_{lj}$$

$$\alpha_{pkl} = \lambda^{(k)} \delta_{kl}$$

$$\alpha_{ij} = \sum_k T_{ki} \lambda^{(k)} T_{kj}$$

$$\alpha_{ij} = \sum_k \lambda^{(k)} x_i^{(k)} x_j^{(k)}$$

We see that the matrix elements of an operator can be expanded in terms of the eigenvalues and eigenvectors of the operator.

$$\alpha^2 X = \lambda X$$

$$\alpha(\alpha X) = \alpha(\lambda X) = \lambda^2 X$$

$$(\alpha^2)_{ij} = \sum_k (\lambda^{(k)})^2 X_i^k X_j^k$$

In general

$$[f(\alpha)]_{ij} = \sum_k X_i^k X_j^k f(\lambda^k)$$

Orthogonality of components.

$$T_{ki} = X_i^k$$

$$I = T^{-1} I T$$

$$\delta_{ij} = \sum_k \sum_l (T^{-1})_{ik} I_{kl} T_{lj}$$

$$= \sum_k T_{ki} T_{kj}$$

$$\delta_{ij} = \sum_k X_i^{(k)} X_j^{(k)}$$

Consider the function

$$\Lambda = \sum_i \sum_j \alpha_{ij} Y_i Y_j / \sum Y_i^2$$

This function will have a minimum

if $\frac{1}{\lambda}\frac{\partial \lambda}{\partial Y_K} = 0$

$$= \frac{1}{N} 2 \sum_j a_{Kj} Y_j - \frac{1}{D} 2 Y_K$$

Hence

$$\sum_j a_{Kj} Y_j = \lambda Y_K$$

The minimum values of λ are simply the eigen values of a and the functions Y are the eigen vectors of a.

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Problem: Diagonalize

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

2's
0
0
2
-2

I Perturbation Theory

Suppose we have solved the equation

$$M V_m = \mu_m V_m$$

We can find an approximate solution of the equation

$$L \phi = \lambda \phi$$

where $L = M + N$ and N is small

in a sense which we shall define later.

Let $\phi = \sum_n c_n v_n$. The equation becomes.

$$(M+N)\phi = \lambda\phi$$

$$\lambda \sum_n c_n v_n = \sum_n c_n M v_n + \sum_n c_n N v_n$$

" $= \sum_n c_n \mu_n v_n$ + "

$$\sum_n (\lambda - \mu_n) c_n v_n = \sum_n c_n N v_n$$

Multiply by v_m and integrate

$$c_m (\lambda - \mu_m) = \sum_n c_n \int v_m N v_n \, d\tau$$

Denote $N_{mn} \equiv \int v_m N v_n \, d\tau$

$$\sum_n c_n N_{mn} + c_m (\mu_m - \lambda) = 0$$

$$\sum_n (N_{mn} + \delta_{mn}(\mu_n - \lambda)) c_n = 0$$

For nontrivial solutions

$$\text{Det} \, | N_{mn} + (\mu_n - \lambda)\delta_{mn} | = 0$$

This determines the eigenvalues λ. The determinant however is infinite and can only be solved approximately in any real case. We shall now discuss such an approximate solution.

Suppose $|N_{mm}| \ll |\mu_m - \mu_n|$ $n \neq m$ The reason for this condition will become clear from what follows. Suppose also that $c_p \approx 1$ and all other c's are small. We have

$$\lambda c_p = \mu_p c_p + \sum_n N_{pn} c_n$$

$$\lambda \approx \mu_p + N_{pp} + \sum_{\substack{n \\ n \neq p}} N_{pn} c_n .$$

To first order

$$\lambda^{(p)} = \mu_p + N_{pp}$$

For $m \neq p$

$$\lambda C_m = \mu_m C_m + \sum_n N_{mn} C_n$$

$$(\mu_p + N_{pp}) C_m = \mu_m C_m + N_{mp} C_p + \sum_{n \neq p} N_{mn} C_n$$

Taking only the first order terms:

$$\mu_p C_m = \mu_m C_m + N_{mp}$$

$$C_m = N_{mp} / (\mu_p - \mu_m) \qquad m \neq p$$

$$\psi_p = V_p + \sum_{m \neq p} \frac{N_{mp}}{(\mu_p - \mu_m)} V_m$$

The ψ's are orthonormal to second order.

The second order $\lambda^{(p)}$ is:

$$\lambda^{(p)} = \mu_p + N_{pp} + \sum_{n \neq p} \frac{N_{pn} N_{np}}{\mu_p - \mu_n}$$

The above treatment assumes there is no degeneracy, that is $\mu_p \neq \mu_n$ where $p \neq n$

Problem of a vibrating non-uniform string of length π.

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = +T\frac{\partial^2 u}{\partial x^2}$$

$$u = V e^{i\omega t}$$

$$\omega^2 V = -T/\rho(x)\frac{d^2V}{dx^2}$$

$$L = -\frac{T}{\rho(x)}\frac{d^2}{dx^2}$$

$$M = -\frac{T}{\rho_0(x)}\frac{d^2}{dx^2}$$

$$N = L - M = \left(-\frac{T}{\rho(x)} + \frac{T}{\rho_0(x)}\right)\frac{d^2}{dx^2}$$

Take $T = \rho_0$

$$N = \left(-\frac{\rho_0}{\rho} + 1\right)\frac{d^2}{dx^2}$$

The unperturbed modes are

$$\sqrt{\frac{2}{\pi}}\sin nx$$

The first order $\lambda(\omega^2)$ for the lowest mode will be:

Assume $\frac{\rho_0}{\rho} = .95 \quad 0 \le x < \frac{\pi}{2}$

$\frac{\rho_0}{\rho} = 1 \quad \frac{\pi}{2} \le x \le \pi$

$$N_{11} = \int_0^{\pi/2} V_1 N V_1 \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \sin x \left[.05\frac{d^2}{dx^2}\right]\sin x \, dx$$

$$= -\frac{2}{\pi}\frac{\pi}{4}.05 = .025$$

$$\omega^2 = 1 - .025$$

$$\omega = .9875$$

Problem: 1) Find second order $\lambda(\omega^2)$ for lowest mode.

2) Find first order λ for first harmonic.

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Last lecture we found an approximate method for finding the eigenvalues of the equation

$$L u = \lambda u$$

where $L = M + N$. The solutions of the equation $N v = \mu v$ are assumed known and

$$\mu_n - \mu_m >> \int v_m N v_n \, d\tau$$

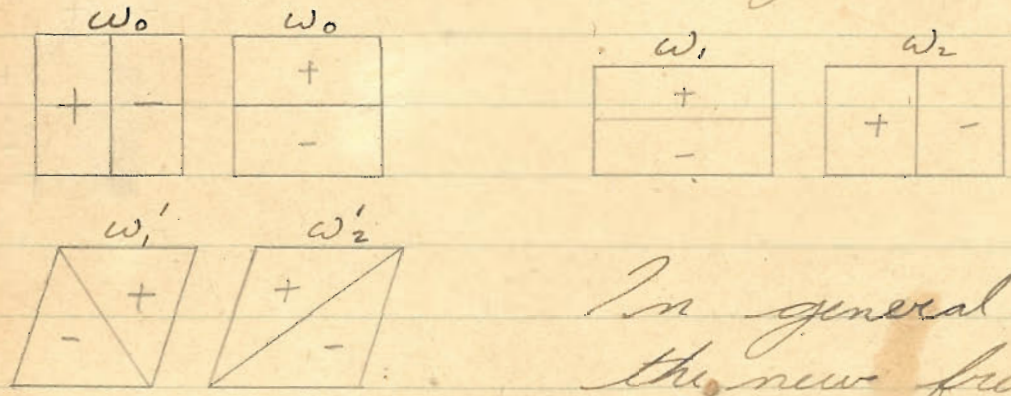
In case $\mu_n = \mu_m$ the method does not apply. We shall now consider this case.

Degenerate Eigenvalues

Suppose esp to the single

eigenvalue ω_0 there are k linearly independent solutions for $V^{(i)}$. If the system is slightly perturbed the degeneracy will in general be removed.

Consider for example the case of a square vibrating membrane



In general the new frequencies and esp normal modes will depend on the type of perturbation

Variational Method

We shall treat the perturbation problem by the variational method.

I Non degenerate case:

$$\lambda_1 = \min \frac{\int f L f \, dx}{\int f^2 \, dx}$$

where $Lu = \lambda u$, and f is any function used to approximate u. As a first approx. let $f = v_0$ where $Mv_0 = \mu_0 v_0$ and $L = M + N$

$$\lambda = \frac{\int v_0 L v_0 dx}{\int v_0^2 dx} = \int v_0 (M+N) v_0 dx$$

$$= \mu_0 + \int v_0 N v_0 dx$$

2) Degenerate case: To μ esp k eigenfunctions $v^{(i)}$. Let:

$$f = \sum_i^k a_i v^i$$

$$\lambda = \int \Sigma a_i v^i (M+N) \Sigma a_i v^i dx \Big/ \int [\Sigma a_i v^i]^2 dx$$

$$= \Big[\int \Sigma a_i v^i M \Sigma a_i v^i dx + \int \Sigma a_i v^i N \Sigma a_i v^i dx \Big] \Big/ \Sigma a_i^2$$

$$= \Big[\mu \int [\Sigma a_i v^i]^2 dx + \sum_{ij} a_i a_j \int v^i N v^j dx \Big] \Big/ \Sigma a_i^2$$

$$= \Big\{ \mu \sum_i a_i^2 - \sum_{ij} a_i a_j N_{ij} \Big\} \Big/ \sum_i a_i^2$$

$$\lambda - \mu = \sum_{ij} a_i a_j N_{ij} \Big/ \sum_i a_i^2 = \Delta\lambda$$

We want to choose the a's so that $\Delta\lambda$ is a minimum.

Let: $$F = \sum_{ij} a_i a_j N_{ij} - \lambda \sum_j a_j^2$$

$$\sum_i a_j^2 = {}^{ij} A$$

$$dF = \sum_j \left\{ \sum_i (N_{ij} + N_{ji} - 2\lambda\delta_{ij}) a_i \right\} da_j = 0$$

$$\sum_i (N_{ij} + N_{ji} - 2\lambda\delta_{ij}) a_i = 0$$

$$\text{Det} \left| N_{ij} + N_{ji} - 2\lambda\delta_{ij} \right| = 0$$

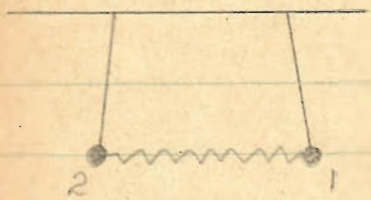
The values of λ which satisfy this determinant are just the minimum values of $\Delta\lambda$ we are seeking. {The value of A is chosen arbitrarily equal to 1}

If $N_{ij} = N_{ji}$

$$\text{Det} \left| N_{ij} - \Delta\lambda\, \delta_{ij} \right| = 0$$

Illustration:

1) Coupled pendulum



$$N_{11} = N_{22} = -N$$
$$N_{12} = N_{21} = +N$$

$$\begin{vmatrix} -N-\Delta\lambda & N \\ N & -N-\Delta\lambda \end{vmatrix} = 0$$

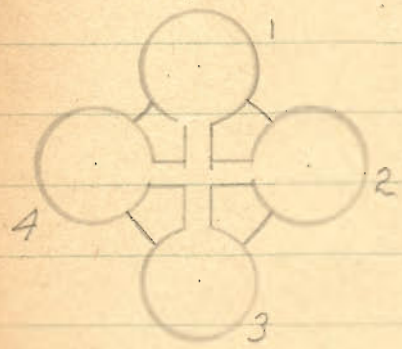
$$N^2 + 2\Delta\lambda N + (\Delta\lambda)^2 - N^2 = 0$$
$$\Delta\lambda(\Delta\lambda + 2N) = 0$$
$$\Delta\lambda^{(1)} = 0$$
$$\Delta\lambda^{(2)} = -2N$$

$$-(\Delta\lambda + N)a_1 + Na_2 = 0$$
$$Na_1 - (\Delta\lambda + N)a_2$$

$\Delta\lambda = 0 \qquad a_1 = +a_2$

$\Delta\lambda = -2N \qquad a_1 = -a_2$

2) Coupled Cavity • Resonators



$$N_{12} = N_{23} = N_{34} = N_{41} = N$$
$$N_{21} \quad N_{32} \quad N_{43} \quad N_{14}$$

$$N_{11} = N_{22} = N_{33} = N_{44} = 0$$

$$\begin{vmatrix} -\Delta\lambda & N & 0 & N \\ N & -\Delta\lambda & N & 0 \\ 0 & N & -\Delta\lambda & N \\ N & 0 & N & -\Delta\lambda \end{vmatrix} = 0$$

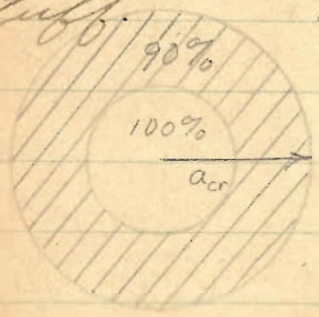
$$\Delta\lambda = 0, 0, +2N, -2N$$

x_{ij} Normal Modes $=$

$$\left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2N \\ 1 & -1 & 1 & -1 & -2N \\ -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{array}\right|$$

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Problem: Cadgit of 90 and 100% stuff. Compute % change in the critical radius



$$\nabla^2 n + S(r)\, n = 0$$

$S = 1 \quad r < \frac{1}{2} a$

$S = .9 \quad \frac{1}{2} a < r < a$

$n = 0 \quad r = a$

To solve as an eigenvalue problem introduce a scale factor $\sqrt{\lambda}$ such that $r = \sqrt{\lambda}\, r'$. Then solve

$$\frac{1}{S(r)} \nabla^2 n = -\lambda n$$

$S(r') = 1 \quad r' < \frac{1}{2}$

$S(r) = .9 \quad \frac{1}{2} < r' < 1$

$n = 0 \quad r =$

$$a_{cr} = \sqrt{\lambda}$$

I Calculus of Variations

A <u>functional</u> is a number which depends on a function.

An example of a functional would be:

$$I = \int f(x)\, L\, f(x)\, dx \Big/ \int f^2(x)\, dx$$

The functions which minimize I have been shown to be the eigenfunctions of the equation

$$L\,u = \lambda\, u$$

Many problems in physics may be reduced to the problem of finding the functions which maximize a functional.

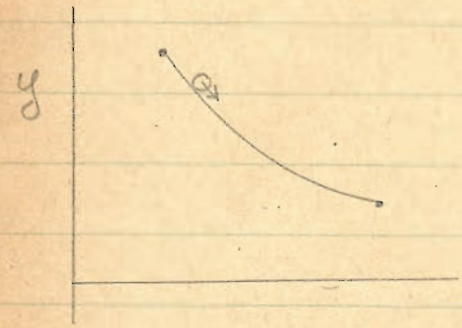
Optics $\int_a^b \frac{ds}{v} = t$ an extremum

Mechanics $\int_{t_1}^{t_2} L\, dt =$ an extremum

The variational method solves these problems of finding the extreme values of functionals.

As an illustration of the method, let us solve the classical

problem of finding the equation of the path between two fixed points (a, b) for which the time it takes a particle to get from a to b under the force of gravity is a minimum.



$$\text{time} = \int_0^{'} \frac{ds}{v}$$

$$\frac{mv^2}{2} = mgy$$

$$v = \sqrt{2gy}$$

$$\text{time} = \int_0^{'} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx$$

We wish to minimize the time. The minimum occurs where the integral is stationary to first order.

$$I = \int \frac{\sqrt{1+y'^2}\, dx}{\sqrt{y}}$$

At minimum

$$\delta I = 0$$

We can find δI in two ways

1) $I + \delta I = \int_0^1 \frac{\sqrt{1+[\frac{d}{dx}(y+\delta y)]^2}}{\sqrt{y+\delta y}}$

$= \int_0^1 dx \sqrt{1+y'^2}\left(1+\frac{y'\delta y'}{1+y'^2}\right) \Big/ \sqrt{y}\left(1+\frac{1}{2}\frac{\delta y}{y}\right)$

$= \underbrace{\int_0^1 \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx}_{I} + \int_0^1 \frac{\sqrt{1+y'^2}}{\sqrt{y}}\left(\frac{y'}{1+y'^2}\delta y' - \frac{1}{2}\frac{\delta y}{y}\right)dx$

$$\delta I = \int_0^1 \frac{\sqrt{1+y'^2}}{\sqrt{y}}\left(\frac{y'}{1+y'^2}\delta y' - \frac{1}{2}\frac{\delta y}{y}\right)dx$$

2) Second method: differentiation

$$\delta I = \int_0^1 \frac{1}{\sqrt{y}}\left(\sqrt{y}\,\tfrac{1}{2}(1+y'^2)^{-\frac{1}{2}}2y'\delta y' - \sqrt{1+y'^2}\,\tfrac{1}{2}y^{-\frac{1}{2}}\delta y\right)dx$$

To get the coefficient of δy integrate by parts:

$$\delta I = \frac{y'\delta y}{\sqrt{y}\sqrt{1+y'^2}}\Big|_0^1 - \int_0^1 \delta y\left[\frac{d}{dx}\left(\frac{y'}{\sqrt{y}\sqrt{1+y'^2}}\right)\right]dx - \int_0^1 \frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}}\delta y\,dx$$

$\delta y = 0$ at $x = 0$ and 1

$\delta I = 0$ Hence (coefficient of δy) $= 0$.

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{y}\sqrt{1+y'^2}}\right) + \frac{\sqrt{1+y'^2}}{2\,y^{3/2}} = 0$$

This is the equation of a cycloid.

In general, if we have

$$I = \int_a^b L(p,y)\,dx$$

where $p = y'$

$$\delta I = \int_a^b \left(\frac{\partial L}{\partial p}\delta p + \frac{\partial L}{\partial y}\delta y\right)dx$$

$$= \cancel{\frac{\partial L}{\partial p}\delta y\Big|_a^b} - \int_a^b \left(\frac{d}{dx}\frac{\partial L}{\partial p} - \frac{\partial L}{\partial y}\right)\delta y\,dx$$

$\delta y = 0$ at a and b

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

This is just the Lagrange equation.

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I Applications of the Variational method

1) Hamiltons Principle

$$\delta \int_0^t L(\dot{x}x)dt = 0$$

where $L = \frac{m}{2}\dot{x}^2 - V(x)$. We shall show that this implies the classical equations of motion

$$\delta \int_0^t L dt = \delta \int_0^t \left[\frac{m}{2}\dot{x}^2 - V(x)\right]dt$$

$$= \int_0^t \left[m\dot{x}\,\delta\dot{x} - \frac{\partial V}{\partial x}\delta(x)\right]dt$$

$$= \cancel{m\dot{x}\delta x\Big|_0^t} - \int_0^t \left[m\ddot{x} + \frac{\partial V}{\partial x}\right]\delta x\,dt = 0$$

Hence: $m\ddot{x} = -\frac{\partial V}{\partial x}$

In performing the variation, we must keep the end points of the path at times 0 and t fixed in space.

2) Electrostatics. We shall derive the Laplace equation from the knowledge that the energy

of the electrostatic field must be an extremum

$$E = \frac{1}{8\pi}\iiint\left[\left(\frac{\partial V}{\partial x}\right)^2 + \ldots\right] dx\,dy\,dz$$

$$\delta E = \frac{1}{8\pi}\iiint\left[2\frac{\partial V}{\partial x}\frac{\partial \delta V}{\partial x} + \ldots\right] dx\,dy\,dz$$

$$2\iint\left\{\int \frac{\partial V}{\partial x}\frac{\partial \delta V}{\partial x}\,dx\right\} dy\,dz$$

$$= 2\iint\left\{\frac{\partial V}{\partial x}\delta V - \int \frac{\partial^2 V}{\partial x^2}\delta V\,dx\right\} dy\,dz$$

The integral is to be taken over the space between conductors. On the conductors δV is assumed zero, hence the integrated part vanishes

$$\delta E = \frac{1}{4\pi}\iiint\left[\nabla^2 V\right]\delta V\,dx\,dy\,dz = 0$$

$$\nabla^2 V = 0$$

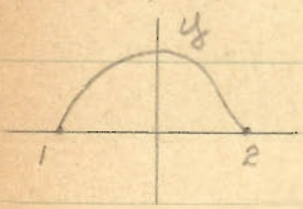
Problem: Derive Poisson's equation.

II Variations Subject to restriction conditions. We wish to minimize $A = A(x_i)$ subject to k restrictions on the x's given by $\phi_i(x_i) = \text{const.}$ To do this we introduce k arbitrary constants, Lagange Multipliers, and minimize the function

$$A + \lambda_1 \phi_1 + \dots \lambda_k \phi_k$$

in the usual way. The k λ's give us the freedom we need to carry out the necessary variations.

As an example, consider the problem of finding the curve of length L under which the area is a maximum.



$$A = \int_1^2 y\,dx$$

$$L = \int_1^2 \sqrt{1 + y'^2}\,dx = \text{const.}$$

$$I = \int [y + \lambda \sqrt{1+y'^2}]\, dx$$

$$\delta I = \int [\delta y + \lambda (1+y'^2)^{-\frac{1}{2}} y' \delta y']\, dx$$

$$= \cancel{\lambda (1+y'^2)^{-\frac{1}{2}} y' \delta y \Big|_1^2}$$

$$- \int_1^2 \left[\frac{d}{dx}(\lambda (1+y'^2)^{-\frac{1}{2}} y') - 1\right] \delta y \, dx = 0$$

$$\frac{1}{\lambda} = \frac{(1+y'^2)^{\frac{1}{2}} y'' - y'(1+y'^2)^{-\frac{1}{2}} y' y''}{(1+y'^2)}$$

$$\frac{1}{\lambda} = \frac{y''}{(1+y'^2)^{\frac{3}{2}}} = 1/\rho$$

This is the equation of a circle of radius $\rho = \lambda$

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Problem: Find the equation of a freely hanging chain by minimizing

$$\bar{y} = \int y \sqrt{1+y'^2}\, dx$$

with

$$l = \int \sqrt{1+y'^2}\, dx$$

Problem: Find the equations of motion arising from a Lagrangian of the form

$$L = \frac{m\dot{x}^2}{2} - V(x) + \frac{k\ddot{x}^2}{2}$$

Show there exists a constant of the motion

$$E = \frac{m\dot{x}^2}{2} + V(x) + \frac{k}{2}(\ddot{x}^2 - 2\dot{x}\dddot{x})$$

In general, if adding a constant to the time in the equations of motion leaves them unchanged there will be a constant of motion analogous to the energy.

if adding a constant to the coordinate leaves the equations unchanged the momentum will be a constant of the motion.

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I Theory of Probability

If A and B are mutually exclusive events (A and B can not occur similtaneously)

$$p[A \text{ or } B] = p[A] + p[B]$$

If A and B are not mutually exclusive

$$p[A \text{ or } B] = p[A] + p[B] - p[A \text{ and } B]$$

If the chance for B do not depend on whether A has happened and visa versa, A and B are said to be uncorrilated

$$p[A \text{ and } B] = p[A]\,p[B]$$

Number of ways of ordering n

distinct quantities

$$= n(n-1)(n-2)\ldots = n!$$

No. of orders n quantities taken m at a time

$$= \underset{1}{n}(\underset{2}{n-1})\ldots(\underset{m}{n-m+1})$$

$$= \frac{n!}{(n-m)!}$$

Suppose we have m A's, n B's & C's etc. The number of ways of ordering the A, B, C, etc. is

$$= \frac{(m+n+l+\ldots)!}{m!\,n!\,l!\ldots}$$

Example: Probability of a full house in poker.

Total no. of ordered hands

$$\frac{52!}{(52-5)!} = 52\cdot 51\cdot 50\cdot 49\cdot 48$$

No. of hands of form AAABB

$AAABB \sim 52 \cdot 3 \cdot 2 \cdot 48 \cdot 3$

No of ways of ordering A, A, A, B, B

$$\frac{5!}{3!\,2!} = 10$$

Total number of full houses / total no. of ordered hands

$$= \frac{52 \cdot 3 \cdot 2 \cdot 48 \cdot 3 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

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Conditional Probability

$$p[A \text{ and } B] = p[A] \times p[B \text{ if } A \text{ has occurred}]$$

$$= p[B] \times p[A \text{ if } B \text{ has occurred}]$$

Problem: An individual burglar has a $1/5$ chance of opening a safe. What is the probability that if

If burglar try a safe it will be opened.

$P[\text{opened on } 1^{st} \text{ trial}] = \frac{1}{5}$

$P[\text{" " } 2^{nd} \text{ "}] = \frac{4}{5} \cdot \frac{1}{5}$

$P[\text{" " } 3^{rd} \text{ "}] = \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5}$

$P[\text{" " } 4^{th} \text{ "}] = \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5}$

$P[\text{opened on one of 4 trials}]$

$$= \frac{1}{5}\left[1 + \frac{4}{5} + \frac{4}{5}^2 + \frac{4}{5}^3\right] = \frac{1}{5}\left(\frac{1-\left(\frac{4}{5}\right)^4}{1-\frac{4}{5}}\right)$$

$$= 1 - \left(\frac{4}{5}\right)^4$$

Easier method: Compute prob safe is unopened in 4 trials.

$$= \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} = \left(\frac{4}{5}\right)^4$$

Prob it is opened $= 1 - \left(\frac{4}{5}\right)^4$.

Expectation:

$R_A \equiv$ return if event A succeeds.

$p[A] \equiv$ probability of event A

Expected Return $= R_A p[A] + R_B p[B] + \ldots$

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To play a game we draw four cards from a deck of 52. 1 Ace pays \$1

2 Aces pay \$16

3 " " \$500

4 " " \$10000

Its cost \$1 to play. Will we make or lose money?

We use the formula for the expected return. We need the probabilities of drawing 1 2 3 4 Aces.

$$\text{Prob one Ace} = \frac{4!}{1!\,3!} \cdot \frac{4}{52} \cdot \frac{48}{51} \cdot \frac{47}{50} \cdot \frac{46}{49}$$

$$= .256$$

Prob. 2 Aces $= \frac{4!}{2!2!} \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{48}{50} \cdot \frac{47}{49} = .0245$

Prob 3 Aces $= \frac{4!}{3!1!} \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{48}{49} = .000695$

Prob 4 Aces $= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} = .00000362$

Expected Return

$1 \times .256$	=	.256
$10 \times .0245$	=	.245
$500 \times .000695$	=	.348
$10000 \times .00000362$	=	.036
		$.885

The House makes money 11¢.

Average Value

$$\bar{N} = \frac{1}{n} \sum_{i}^{n} N_i n_i$$

$\frac{n_i}{n}$ = Prob of N_i

$$\bar{N} = \sum_{i}^{n} N_i p_i = \text{Expected Return}$$

Suppose in a single trail the probability of success is p. and failure $q \equiv 1 - p$. If we make N trails, what is the probability of exactly K successes?

$P_N(K) \equiv$ Prob K successes in N trails

$= P_{N-1}(K-1) \cdot p + P_{N-1}(K) \cdot q$

Introduce $F_N(t) = \sum_K t^K P_N(K)$

$F_N(t) = p \cdot t F_{N-1}(t) + q \cdot F_{N-1}(t)$

$F_N(t) = (pt + q) F_{N-1}$

$F_N(t) = (q + pt)^N F_0(t)$

But $F_1(t) = q + pt = (q + pt) F_0(t)$

$F_0(t) = 1$

$$F_N(t) = (q + pt)^N$$

$$= \sum_K \left\{ \frac{N!}{(N-K)!\,K!}\, q^{N-K} p^K \right\} t^K$$

$$P_N(K) = \frac{N!}{(N-K)!\,K!}\, q^{N-K} p^K$$

Second method computing $P_N(K)$

Prob of K successes in N trial in a given order.

$$\overbrace{p \cdot p \cdots p}^{K} \overbrace{q \cdot q \cdots q}^{N-K} = p^K q^{N-K}$$

Number of possible orders.

$$= \frac{N!}{(N-K)!\,K!}$$

$$P_N(K) = \frac{N!}{(N-K)!\,K!} \cdot p^K q^{N-K}$$

Problem: Expand using Stirlings approx. Show max. at $K = pN$ and approx Gaussian for large K

5/19/47

$P_N(K)$ = prob K successes in N trails.

The average number of successes is:

$$\bar{K} = \sum_K K P_N(K)$$

We can find $\bar{K}$ in terms of the generating function.

$$F_N(t) = \sum_K t^K P_N(K)$$

$$F_N'(t) = \sum t^{K-1} K P_N(K)$$

$$F_N'(1) = \bar{K}$$

But $F_N(t) = (q+pt)^N$

$F_N'(t) = Np(q+pt)^{N-1}$

$$F_N'(1) = Np = \bar{K}$$

R.M.S Deviation

$K - Np$ or $K - \bar{K}$ is the deviation from the mean. The ave deviation is obviously zero.

The r.m.s. deviation, $[\overline{(K-\bar{K})^2}]^{1/2}$, is the square root of the average of the square of the deviation from the mean.

$$\overline{(K-\bar{K})^2} = \sum_K (K-\bar{K})^2 P(K)$$

$$= \sum K^2 P(K) - 2\bar{K}\sum K P(K) + \bar{K}^2 \sum P(K)$$

$$= \overline{K^2} - 2(\bar{K})^2 + (\bar{K})^2$$

$$\overline{(K-\bar{K})^2} = \overline{K^2} - (\bar{K})^2$$

This formula gives a simple method of finding the r.m.s. deviation if we can find $\overline{K^2}$.

We can find $\overline{K^2}$ in terms of the generating function.

$$\overline{K^2} = \sum_K K^2 P(K)$$

$$F_N(t) = \sum P_N(K)\, t^K$$

$$t\, F_N'(t) = \sum K P_N(K)\, t^K$$

$$\frac{d}{dt}\left[t\, F_N'(t)\right] = \sum K^2 P(K)\, t^{K-1}$$

$$F_N'(t) + t F_N''(t) = \sum_K K^2 P(K)\, t^{K-1}$$

$$F_N'(1) + F_N''(1) = \overline{K^2}$$

$$F_N(t) = (q + pt)^N$$

$$F_N'(t) = Np(q+pt)^{N-1}$$

$$F_N''(t) = p^2 N(N-1)(q+pt)^{N-2}$$

$$F_N'(1) = pN$$

$$F_N''(1) = p^2N^2 - p^2N$$

$$\overline{K^2} = p^2N^2 + pN - p^2N$$

$$\overline{(K-\bar{K})^2} = \overline{K^2} - \bar{K}^2$$

$$= pN - p^2N = Np(1-p)$$

R. M. S. deviation

$$= \sqrt{Npq} = \sqrt{\bar{K} \cdot q}$$

Percentage R.M.S. deviation

$$= \frac{\sqrt{Npq}}{Np} = \frac{1}{\sqrt{N}}\sqrt{\frac{q}{p}}$$

Approximate Expression for $P_N(K)$ for large N

$$P_N(K) = \frac{N!}{(N-K)!\,K!}\, p^K q^{N-K}$$

$$N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$$

Let the fraction of success be $f = K/N$. We have

$$P_N(K) = \frac{\sqrt{2\pi N}\left(\frac{N}{e}\right)^N p^{Nf} q^{N(1-f)}}{\sqrt{2\pi Nf}\left(\frac{Nf}{e}\right)^{Nf}\sqrt{2\pi N(1-f)}\left(\frac{N(1-f)}{e}\right)^{N(1-f)}}$$

$$= \frac{1}{\sqrt{2\pi Nf(1-f)}}\left[\left(\frac{p}{f}\right)^f\left(\frac{q}{1-f}\right)^{1-f}\right]^N$$

To find the max of $\left[\left(\frac{p}{f}\right)^f\left(\frac{q}{1-f}\right)^{1-f}\right]$ as a function of f, take ln derivative and equate it to zero.

$$\frac{d}{df} \ln\left[\left(\frac{p}{f}\right)^{f}\left(\frac{q}{1-f}\right)^{1-f}\right]$$

$$= \frac{d}{df}\{f\ln p - f\ln f + (1-f)\ln q - (1-f)\ln(1-f)\}$$

$$= \ln p - \ln f - 1 - \ln q + \ln(1-f) + 1$$

$$= 0 \qquad \text{Hence} \quad p = f_{max}$$

To get an approximate form for $\ln[\ ]$ we expand about the maximum.

$$\ln[f] = \ln[p] + (f-p)\frac{d}{df}\ln[\ ]\Big|_{f=p} + \frac{1}{2}(f-p)^2\frac{d^2}{df^2}\ln[\ ]\Big|_{f=p} + \ldots$$

$$= 0 + 0 + \frac{(f-p)^2}{2}\left(-\frac{1}{pq}\right)$$

We obtain

$$[\ ]^N \approx e^{N\left(-\frac{(f-p)^2}{2pq}\right)}$$

$$P_N(K) \approx \frac{1}{\sqrt{2\pi N p q}}\, e^{-\frac{N}{2pq}(f-p)^2}$$

If we define the width of the curve as the width at $e^{-\frac{1}{2}}$ we obtain

$$\tfrac{1}{2}\Delta w = (f_1 - p) = \pm\sqrt{\frac{pq}{N}}$$

Problems: I Suppose one has 52 balls numbered 1 to 52. The balls are drawn one at a time from a hat. What is the probability that no number on a ball corresponds to its number in the sequence of draws.

II One man has K pennies, another has N. What is the probability that in a game of matching pennies the man with K pennies will be cleaned out.

III How should one bet $50 at roulette to have the max.

probability of winning $10.

IV Throw a die. If you hit 21 you win $5. If you go over 21 you pay $5. If you stop short of 21 you pay $(21 - x). At what x should you stop

3/21/47

Probability density.

If the number whose probability we seek can take on a continuum of values, then the probability of a particular value will be zero. However, the probability that the number has a value in a range dx about a particular x is finite and is given by:

$$P(x)dx$$

$P(x)$ is the probability density. The prob. of a number between a & b is:

$$\int_a^b P(x)\,dx$$

If we integrate over all possible values of x we should get 1

$$\int_{all\,x} P(x)\,dx = 1$$

In two dimensions, we have

$$P(x,y)\,dx\,dy$$

$$p(x)\,dx = \left\{\int_{all\,y} P(x,y)\,dy\right\} dx$$

$$\int\int_{all\,x\,\&\,y} P(x,y)\,dx\,dy = 1$$

Problem: Find the interval about the maximum of a gaussian that contains half the total area. i.e. Find the probable error.

5/23/47

X_1 has prob density $f_1(x_1)$
X_2 " " " $f_2(x_2)$

If $y = x_1 + x_2$ the prob density of y is

$$P(y) = \int f_1(x) f_2(y-x)\, dx$$

In general

$$z = \sum_i x_i$$

where $x_i \sim f_i(x_i)$

$$P(z)dz = \int\int \cdots f_1(x_1) f_2(x_2 - x_1) f_3(x_3 - x_2) \cdots dx_1 dx_2 dx_3 \cdots dz$$

If we use the Fourier transforms of the probability densities

$$\phi_i(\omega) = T f_i(x_i)$$

$$\Phi(\omega) = T P(z)$$

$$\Phi(\omega) = \phi_1(\omega)\phi_2(\omega)\phi_3(\omega)\cdots$$

Moments, M_n

$$1^{st} = \bar{x} = \int x\, p(x)\, dx$$

$$n^{th} = \overline{x^n} = \int x^n p(x)\, dx$$

$$0^{th} = 1 = \int p(x)\, dx$$

A knowledge of all the moments of a prob density enables us to find the distribution

$$\phi(\omega) = \int e^{i\omega x} p(x)\,dx$$

$$= \int \sum_n \frac{(i\omega x)^n}{n!} p(x)\,dx$$

$$= \sum_n \frac{(i\omega)^n}{n!} \int x^n p(x)\,dx$$

$$= \sum_n \frac{(i\omega)^n}{n!} M_n$$

$$p(x) = \frac{1}{2\pi} \int e^{-i\omega x} \phi(\omega)\,d\omega$$

$$= \frac{1}{2\pi} \int e^{-i\omega x} \sum_n \frac{(i\omega)^n}{n!} M_n$$

Problems

1) Prove $\int P(y)\,dy = 1$

$y = x_1 + x_2$ and $\int_{f_2(x_2)\,dx_2}^{f_1(x_1)\,dx_1} = 1$

2) Prove $\bar{y} = \bar{x}_1 + \bar{x}_2$

Probability distribution arising from a large number of small

errors superimposed on one another. We shall assume that all causes of error have zero mean value. If this is not the case, it will not change the shape of the curve but only shift the maximum.

$$\Phi(\omega) = \phi_1(\omega)\,\phi_2(\omega)\ldots$$
$$= \left(1 + i\omega M_1 + \frac{(i\omega)^2}{2!}M_2' + \ldots\right)\left(1 + i\omega M_1' + \frac{(i\omega)^2}{2!}M_2'' + \ldots\right)\ldots$$

We have assumed $M_1 = 0 \;:\; M_{2k+1} = 0$

$$= \left(1 - \frac{\omega^2}{2}\overline{X_1^2} + \ldots\right)\left(1 - \frac{\omega^2}{2}\overline{X_2^2} + \ldots\right)\ldots$$

$$\ln \Phi = \sum \ln \phi_i = \left(-\frac{\omega^2}{2}\overline{X_1^2} + \ldots\right) + \left(-\frac{\omega^2}{2}\overline{X_2^2} + \ldots\right) + \ldots$$

$$= -\frac{\omega^2}{2}\sum_i \overline{X_i^2} + O\left[\omega^4 \sum_i \overline{X^4}\right] +$$

Suppose there are N variables and that $\overline{X_i^2} \approx a^2$

$$= \frac{\omega^2 a^2 N}{2} + O\left[\frac{(\omega^2 a^2 N)^2}{2N}\right] +$$

We neglect all but the first term

for large N

$$\Phi(\omega) = e^{-\frac{\omega^2}{2}\sum_i \overline{x_i^2}}$$

$$P(z) = \frac{1}{2\pi}\int \Phi(\omega) e^{-i\omega z} d\omega$$

$$= \frac{1}{\sqrt{2\pi\Sigma_i}} e^{-\frac{z^2}{2\Sigma}}$$

where $\sum_i \overline{x_i^2} = \overline{z^2} = \sigma^2$

[Problem: prove $\overline{z^2} = \sum_i \overline{x_i^2}$]

σ is the standard deviation.

$$P(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}}$$

The probability that the error lies in the range $\pm r\sigma$ is

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-r\sigma}^{r\sigma} e^{-\frac{z^2}{2\sigma^2}} dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-r}^{r} e^{-\frac{y^2}{2}} dy$$

This is 50% for the range $\pm .67\sigma$

This distribution is not to be assumed correct too far from the maximum ($\sim r > 3$). There is always a small probability of large errors and we have not included this possibility in our derivation.

Second derivation of the Gaussian

Assume all variables have the same distribution. Let n be the number of variables included in the distribution

$$P_{n+1}(y) = \int P_n(z) f(y-z)\, dz$$

$$= \int P(y-x) f(x)\, dx$$

Expand in a Taylor series about y.

$$P_{n+1}(y) = \int P_n(y) f(x)\, dx_i - \int P_n'(y)\, x_i f(x)\, dx_i + \int P_n''(y) \frac{x_i^2}{2!} f(x)\, dx_i + \ldots$$

Assuming $M_1 = 0$

$$P_{n+1}(y) = P_n(y) + \frac{1}{2} P_n''(y) \overline{X_i^2} + \dots$$

$$\frac{P_{n+1}(y) - P_n(y)}{1} = \frac{1}{2} \overline{X_i^2} \cdot \frac{\partial^2 P}{\partial y^2} + \dots$$

If we now consider the $P_n(y)$ as a continuous function of n, then we can equate the difference quotient to the derivative to get a partial differential equation.

$$\frac{\partial P_n(y)}{\partial n} = \frac{1}{2} \overline{X_i^2} \frac{\partial^2 P}{\partial y^2}$$

This is the heat flow or diffusion equation. The solution is:

$$P_n(y) = \frac{1}{\sqrt{4\pi \frac{1}{2} n \overline{X_i^2}}} e^{-\frac{y^2}{4 \frac{1}{2} n \overline{X_i^2}}}$$

$$n \overline{X_i^2} = \sigma^2$$

$$P_n(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$$

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We have seen that if we combine many small errors the resulting distribution is Gaussian in form. We were also careful to point out that the Gaussian being an approx. is often incorrect far from the maximum.

An example of this occurs when we attempt to compute the angular distribution of particles scattered by a foil. In individual collisions the prob of scattering thru θ is $\sim \sin^4 \theta$. In traversing the foil most particles will suffer many small collisions. This leads to a Gaussian for small values of θ. Some particles will large deflections in a single collision. These particles will

be scattered thru large θ and their distribution will be $\frac{1}{\sin\theta}$ and not Gaussian.

Suppose we have the problem of calculating how many rain drops hit a quarter in a time T. The probability of a count in a time interval dT is dT/τ

τ = mean time between counts

$\frac{1}{\tau}$ = mean counting rate

= average no. counts per sec.

We will justify these definitions later.

Let us compute first the probability that no rain drop hits the coin in a time T.

$$P_0(T+dT) = P_0(T) \cdot \text{prob of no count in } dT$$

$$= P_0(T)\left(1 - \frac{dT}{\tau}\right)$$

$$\{P_0(T+dT) - P_0(T)\}/dT = -\frac{1}{\tau} P_0(T)$$

$$\frac{d}{dT} P_0(T) = -\frac{1}{\tau} P_0(T)$$

$$P_0(T) = e^{-T/\tau}$$

We can also arrive at this result by solving the functional equation

$$P_0(T_1 + T_2) = P_0(T_1) P_0(T_2)$$

This can be done in two ways

1) Partial differentiation

$$\frac{d P_0(x)}{dx} P_0(y) = \frac{d P(x+y)}{d(x+y)} \cdot 1$$

$$\frac{d P_0(y)}{dy} P_0(x) = \frac{d P(x+y)}{d(x+y)} \cdot 1$$

$$P_0'(x) P_0(y) = P_0'(y) P_0(x)$$

$$\frac{P_0'(x)}{P_0(x)} = \frac{1}{\tau} = \frac{P_0'(y)}{P_0(y)}$$

$$P_0(T) = e^{-T/\tau}$$

2) Fourier Transforms

$$P_0(V) = P_0(x)\,P_0(V-x)$$

$$\int_{-\infty}^{\infty} e^{i\omega V} P_0(V)\,dV = P_0(x) \int_{-\infty}^{\infty} P_0(V-x)\,e^{i\omega V}\,dV$$

Let $V - x = y \qquad dV = dy$

$$\phi(\omega) = P_0(x)\,e^{i\omega x} \int_{-\infty}^{\infty} P_0(y)\,e^{i\omega y}\,dy$$

$$\phi(\omega) = P_0(x)\,e^{i\omega x}\,\phi(\omega)$$

$$P_0(x) = e^{-i\omega x}$$

$$P_0(dT) = e^{-i\omega dT} = 1 - i\omega dT = 1 - \frac{dT}{\tau}$$

$$i\omega = \frac{1}{\tau}$$

$$P_0(x) = e^{-T/\tau}$$

A third approach to the original problem is to divide the interval T into n sub intervals ΔT, $n\Delta T = T$. The probability of no count in each interval is $(1 - \frac{\Delta T}{\tau})$. The

probability of no count in T is:

$$P_0(T) = \left(1 - \frac{\Delta T}{\tau}\right)\left(1 - \frac{\Delta T}{\tau}\right) \cdots \left(1 - \frac{\Delta T}{\tau}\right)$$

(n terms)

$$P_0(T) = \left(1 - \frac{\Delta T}{\tau}\right)^n$$

$$= \left\{\left(1 - \frac{\Delta T}{\tau}\right)^{-\frac{\tau}{\Delta T}}\right\}^{-\frac{n\Delta T}{\tau}}$$

Take $\lim \Delta T \to 0$

$$= e^{-T/\tau}$$

We now wish to calculate the probability of <u>K counts in the interval T</u>. If we divide our interval into M parts ΔT, then what we want is the probability of K successes in M trials:

$$P_K(M\Delta T) = \frac{M!}{(M-K)!K!} p^K q^{M-K}$$

Let $M \to \infty$ $\quad \frac{M!}{(M-K)!} \to M^K$

$$= \frac{(Mp)^K}{K!} q^M$$

$$= \frac{(M\frac{\Delta T}{\tau})^K}{K!}\left(1 - \frac{\Delta T}{\tau}\right)^M$$

In the limit $= \frac{1}{K!}\left(\frac{T}{\tau}\right)^K P_0(T)$

$$P_K(T) = \frac{(T/\tau)^K}{K!} e^{-T/\tau}$$

This is the Poisson distribution

$T/\tau = \bar{K}$

$$\sum_K P_K(T) = \left\{\sum_K \frac{(T/\tau)^K}{K!}\right\} e^{-T/\tau} = 1$$

$$\bar{K} = \sum_K K P_K(T) = \frac{T}{\tau}\left\{\sum_K \frac{(T/\tau)^{K-1}}{(K-1)!}\right\} e^{-T/\tau}$$

$$\bar{K} = T/\tau$$

Counts per sec, $\bar{K}$/sec $= \frac{1}{\tau}$.

Obviously $\overline{K(K-1)} = \left(\frac{T}{\tau}\right)^2$.

$$\overline{K^2} - \bar{K} = (\bar{K})^2$$

$$\overline{K^2} = (\bar{K})^2 + \bar{K}$$

$$\overline{(K-\bar{K})^2} = \overline{K^2} - \bar{K}^2 = \bar{K}$$

Standard deviation $= \sqrt{\bar{K}}$

Percentage deviation $= \frac{\sqrt{R}}{R} = \frac{1}{\sqrt{R}}$

A second way of getting at $P_K(T)$ is to solve the difference equation

$$P_{K+1}(T+\Delta T) = P_K(T)\cdot\frac{\Delta T}{\tau} + P_{K+1}(T)\cdot\left(1-\frac{\Delta T}{\tau}\right)$$

Problem: $T \gg \tau$ In an interval T we expect $\frac{T}{\tau} = C_0 T$ counts. The measured counting rate is: $C = K/T$. Show:

$$P(C) \cong \frac{1}{\sqrt{2\pi C_0 T}}\, e^{-\frac{(C-C_0)^2}{2C_0 T}}$$

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Solution of the equation

$$P_K(t+\Delta t) = P_{K-1}(t)\frac{\Delta t}{\tau} + P_K\left(1-\frac{\Delta t}{\tau}\right).$$

$$\tau\frac{dP_K}{dt} = P_{K-1} + P_K$$

$$f(x\,t) = \sum x^k P_k(t)$$

$$\tau \sum x^k P_k' = \sum x^k P_{k-1} - \sum x^k P_k$$

$$\tau\, \partial f/\partial t = (x-1) f$$

$$\ln f = (x-1)\frac{t}{\tau} + g(x)$$

$$f = G(x)\, e^{\frac{(x-1)}{\tau} t}$$

$P_k(0) = \delta_{0,k}$ hence $f(x\,0) = 1$

and $G(x) = 1$

$$f(x\,t) = e^{-\frac{t}{\tau}} e^{\frac{x\,t}{\tau}}$$

$$= e^{-t/\tau} \sum \frac{1}{k!}\left(\frac{x\,t}{\tau}\right)^k$$

$$P_k(t) = \frac{e^{-t/\tau}}{k!}\left(\frac{t}{\tau}\right)^k$$

Conditional Probability

$$P(A+B) = P(A) \cdot P(B \text{ if } A)$$

$$= P(B) \cdot P(A \text{ if } B)$$

$$\frac{P(B)}{P(A)} = \frac{P(B \text{ if } A)}{P(A \text{ if } B)}$$

$$P(A \text{ \& } C) = P(A) \cdot P(C \text{ if } A)$$

$$\frac{P(C \text{ if } A)}{P(B \text{ if } A)} = \frac{P(C)}{P(B)} \cdot \frac{P(A \text{ if } C)}{P(A \text{ if } B)}$$

Problem of coins in drawers. There are three drawers. One contains two gold coins, the second one gold and one silver coin, and the third two silver coins. If a drawer is opened and a gold coin picked out at random, what is the probability the other coin in the drawer is gold.

Prob gold if gold = Prob 2 gold ÷ Prob 1 gold.

$$= \frac{1}{3} / \frac{1}{2} = \frac{2}{3}$$

Suppose we perform an experiment to test ~~two~~ theories, B and C. We get a result A.

In the light of this result we wish to know the relative probability of the two theories being correct.

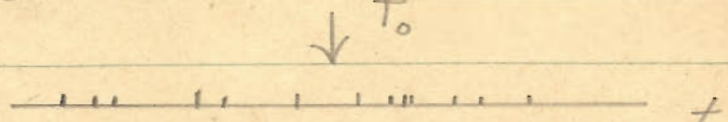
$$\frac{P(C \text{ if } A)}{P(B \text{ if } A)} = \frac{P(C)}{P(B)} \cdot \frac{P(A \text{ if } C)}{P(A \text{ if } B)}$$

A good experiment is one for which the factor $P(A \text{ if } C)/P(A \text{ if } B)$ is extreme. Note that the a priori probability of the two theories, $P(C)/P(B)$ also enters into the problem.

Example. A coin comes up heads 10 times in a row. What is the probability of its being two headed

$$\frac{P(2 \text{ if } 10)}{P(1 \text{ if } 10)} = \frac{P(2)}{P(1)} \cdot \frac{1}{(1/2)^{10}}$$

A cosmic ray counter is counting at an average rate $1/\tau$. We pick some time, T_0; what is the average value of the interval between counts in which T_0 falls.



$$P_0(T) = \frac{1}{\tau} e^{-T/\tau}$$

Prob T_0 in an interval $T = \frac{1}{\tau} T P_0(T)$

Av interval $= \frac{1}{\tau^2} \int_0^\infty T^2 e^{-T/\tau} = 2\tau$

Least Square Method of fitting data.

Suppose we have N measurements $a_1, a_2, \ldots a_N$. What is the prob that the true value of a lies between a_0 and $a_0 + da$

$$P_{\text{After}}(a_0)\, da = C\, P_{\text{a priori}}(a_0) \cdot P(a_1 a_2 a_3 \ldots a_N \text{ if } a_0)\, da$$

$$\text{Prob}(a_1, a_2, a_3 \ldots a_N \text{ if } a_0) = \frac{1}{(\sqrt{2\pi\sigma^2})^N} e^{-\frac{1}{2\sigma^2}\left[N a_0^2 + 2a_0 \Sigma a_i + \Sigma a_i^2\right]}$$

To make this prob. density a maximum, we must minimize the exponent.

$$\frac{d}{da_0}[\;] = 2Na_0 - 2\Sigma a_i = 0$$

$$\bar{a}_0 = \frac{\Sigma a_i}{N}$$

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We have performed a set of measurements $X_1\ X_2\ X_3 \ldots X_N$. We want to know the probability that true value of the quantity measured is a. If the a priori probability of its being "a" was $P_0(a)\,da$, then after the experiments we have:

$$P_{\text{After}}(a)da \sim P_0(a)da \cdot P(X_1, X_2, X_3 \ldots X_n \text{ if } a)$$

Let us assume $P(x_k \text{ if } a) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x_k - a)^2}{2\sigma}}$

Then

$$P(\text{set } x_i \text{ if } a) = \prod_i P(x_i \text{ if } a)$$

$$P_A(a)da \sim \frac{1}{\sigma^N} \exp\left(-\sum_1^N (x_i - a)^2 \frac{1}{2\sigma^2}\right)$$

$$= \frac{1}{\sigma^N} \exp -\left(\sum_1^N x_i^2 - 2a\sum x_i + a^2 N\right)\frac{1}{2\sigma^2}$$

We define $\bar{x} = \frac{1}{N}\sum_i x_i$

$$\overline{x^2} = \frac{1}{N}\sum x_i^2$$

$$= \frac{1}{\sigma^N} \exp -(a - \bar{x})^2 \frac{N}{2\sigma^2} \exp -(\overline{x^2} - (\bar{x})^2)\frac{N}{2\sigma^2}$$

From this we see that the most probable value of "a" is $\bar{x}$. We also see that the curve is a Gaussian with standard duviation

$$\sigma_0 = \sigma / \sqrt{N}$$

We have duvied this result assuming that the σ's for each measurement were the same. If this is not

the case, it can be shown:

$$\text{Most probable } a = \frac{\sum w_i x_i}{\sum w_i}$$

where $w_i \equiv \frac{1}{\sigma_i^2}$. The standard deviation for the combined measurements is

$$\frac{1}{\sigma_0} = \sum w_i = \sum \frac{1}{\sigma_i^2}$$

Suppose we have made a set of measurements and we have the problem of determining not only the best value of "a" but also the best value of σ.

$$P_A(a\sigma)\,da\,d\sigma \sim P_0(a\sigma)\,da\,d\sigma\,\frac{1}{\sigma^N} e^{-N\frac{(\overline{\Delta^2} + (a-\bar{x})^2)}{2\sigma^2}}$$

The best "a"

$$P(a)\,da \sim \left\{\int P_0(a\sigma)\,d\sigma\right\} da$$

$$= da \int_{-\infty}^{\infty} \frac{P_0(a\sigma)}{\sigma^N} e^{\frac{-N\left[\overline{\Delta^2} + (a-\bar{x})^2\right]}{2\sigma^2}} d\sigma$$

In order to integrate this, we introduce a new variable $1/\sigma = X$. We must also make some assumption about the a priori distribution of σ. The assumption usually made is that $1/\sigma$ is distributed uniformly. The result of the integration is:

$$= P(a)da\ F(N)\left\{\overline{\Delta^2} + (a-\bar{x})^2\right\}^{-\left(\frac{N-1}{2}\right)}$$

The "minus one" appearing in the exponent is not to be taken too seriously since the value of the exponent depends on our a priori assumptions about the distribution of σ. The result holds for large N

We can write this in a slightly different way:

$$P_A(a)da \sim \frac{1}{(\overline{\Delta^2})^{\frac{N}{2}}} e^{-\frac{N}{2}\ln\left(1+\frac{(a-\bar{x})^2}{\overline{\Delta^2}}\right)}$$

$$= \frac{1}{(\overline{\Delta^2})^{N/2}} e^{-\frac{N}{2}\frac{(a-\bar{x})^2}{\overline{\Delta^2}}}$$

This is a Gaussian with standard deviation

$$\sigma_0 = \sqrt{\frac{1}{N}(\overline{\Delta^2})}$$

Best σ^2

$$P_A(\sigma)d\sigma \sim \left\{\int P(\sigma a)da\right\}d\sigma$$

$$\sim \frac{1}{\sigma^N} e^{-\frac{N\overline{\Delta^2}}{2\sigma^2}}$$

To find the maximum differentiate the ln.

$$\ln P = -\frac{N\overline{\Delta^2}}{2\sigma^2} - N\ln\sigma$$

$$\frac{N\overline{\Delta^2}}{\sigma^3} - \frac{N}{\sigma} = 0$$

$$\sigma^2 = \overline{\Delta^2}$$

The standard deviation for σ is $\frac{2(\overline{\Delta^2})^2}{N}$

The percent error $\sim \frac{1}{\sqrt{N}}\sqrt{\overline{\Delta^2}}$

Method of Least Squares

Suppose we know one quantity as a function of another: $y_i \sim x_i$. We wish to fit a curve $y = f(x\alpha)$ to the data but a choice of the constants α.

$$\text{error } y \sim \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(y_i - y)^2}{2\sigma_i^2}}$$

$$P_A(\alpha)\,d\alpha \sim P_0(\alpha)\,d\alpha \prod_i e^{-\frac{(y_i - f(x_i,\alpha))^2}{2\sigma_i^2}}$$

$$\sim \exp -\frac{1}{2}\sum_i w_i (y_i - f(x_i,\alpha))^2$$

The best values of α are those which make the sum a minimum.